Lecture 2: 1-D Problems, Linear Analysis, Bifurcation, Bifurcation Diagram

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1 Introduction

Figure 1: Phase portrait of a 1-D problem (first-order problem). Fixed points are shown with the filled and unfilled bullets. Arrowheads show the direction of the flow.

We continue with the one-dimensional problems (flow on a line) given in the following form:

$$\dot{x} = f(x),$$

(1)

where function $f(x)$ can be linear or nonlinear. Figure 1 shows the phase portrait of a 1-D problem.

At times it is beneficial to simplify your problems in order to analyse them. This is especially true in case of problems that are described by nonlinear differential equations. **Linearisation** is a tool for analysing the dynamics of nonlinear systems.

2 Linearisation of 1-D systems

2.1 Linearisation about fixed point $x^*$ and linear stability analysis of $x^*$

Figure 2: Phase portraits of the original flow described by Eq. (1) and its linearisation Eq. (5). Linearisation is done about fixed points $x_1^*$ and $x_2^*$. Linearised solutions are shifted to coincide with the position of the fixed point. Linearised phase portraits are similar to the original flow only in close proximity to their respective fixed points.

Let’s examine the dynamics of Eq. (1) close to its fixed point $x^*$. We assume the solution is in the following form:

$$x(t) = x^* + \eta(t),$$

(2)

where $|\eta| \ll 1$ is a small perturbation. The behaviour and change of solution $x$ over time

$$\dot{x} = (x^* + \eta)' = \dot{\eta}. $$

(3)

At the same time it holds that $\dot{x} = f(x)$. The combination of these results gives

$$\dot{\eta} = f(x) = f(x^* + \eta) = \left[\text{Taylor series expansion about } x^*\right] = f(x^*) + \frac{f'(x^*)}{1!}(x^* + \eta - x^*) + \frac{f''(x^*)}{2!}(x^* + \eta - x^*)^2 + \cdots =$$

$$= \left[f(x^*) = 0\right] = f'(x^*)\eta + \frac{f''(x^*)}{2!}\eta^2 + \cdots \approx f'(x^*)\eta. \quad (4)$$

Higher order terms, $O(\eta^2)$
If $f'(x^*) \neq 0$, then the term $|f'(x^*)\eta| \gg \left|\frac{f''(x^*)}{2!}\eta^2\right|$. Neglecting $O(\eta^2)$ yields the linearisation of the system at fixed point $x^*$

$$\dot{\eta} = s\eta,$$

where $s = f'(x^*)$ is simply the slope of function $f(x)$ at $x^*$. The linearised form (5) of the original system (1) is a familiar equation to us. Solution of Eq. (5) has the form

$$\eta(t) = \eta_0 e^{st},$$

where $\eta_0$ is a constant. The stability of this solution based on its behaviour is the following:

- If $s > 0$, then the solution is exponentially growing (exploding). We say the solution is **unstable**.
- If $s < 0$, then the solution is exponentially decaying. Asymptotically approaching a stable value ($\eta(t) \rightarrow 0$ for $t \rightarrow \infty$). We conclude that the solution is **stable**.

This behaviour is also seen on the phase portrait shown in Fig. 2. The linearised phase portraits (shown in red) preserve the characteristics of the original flow (shown in black) close to their respective fixed point $x_1^*$ or $x_2^*$.

If $s = f'(x^*) = 0$ then no information from linearisation can be obtained. The determination of stability type needs further analysis (additional terms of the Taylor series expansion (4)).

### 2.2 Examples of cases where $f'(x^*) = 0$

![Phase portraits](image)

Figure 3: Phase portraits of Eqs. (7)–(10). Upper row of the graphs feature a new type of fixed point—the **half-stable fixed point**.

Consider the following systems

$$\dot{x} = x^2,$$  

$$\dot{x} = -x^2,$$  

$$\dot{x} = x^3,$$  

$$\dot{x} = -x^3.$$  

Systems (7)–(10) share the same fixed point $x^* = 0$ and have different stability types, this can be clearly seen in Fig. 3. Systems (7) and (8) feature a new type of fixed point—the **half-stable fixed point**. The origin and nature of the half-stable fixed point will become clear below (Sec. 6.1).

Systems (9) and (10) feature an **algebraic decay** $x(t) \sim t^{-\delta}$ near the fixed point (see Slide 3), as opposed to a more common **exponential decay** $x(t) \sim e^{-\gamma t}$. We will revisit this idea in future lectures (algebraic decay may be a possible source of confusion in analysis of 2-D systems).
3 Example of linearisation and linear stability analysis

3.1 Linearisation of Logistic equation

The Logistic equation has the form

$$\dot{x} = rx \left(1 - \frac{x}{K}\right),$$

where \(r > 0\) and \(K > 0\) are the system parameters. Fixed points are

$$\dot{x} = 0 \Rightarrow r x^* \left(1 - \frac{x^*}{K}\right) = 0 \Rightarrow \begin{cases} x_1^* = 0, \\
           x_2^* = K. \end{cases}$$

In order to obtain linearisation in the form (5), we calculate the slope and evaluate it at the fixed point \(x^*\).

The slope is

$$f'(x) = \frac{d}{dx} \left[rx \left(1 - \frac{x}{K}\right)\right] = \frac{d}{dx} \left(rx - \frac{r}{K} x^2\right) = r - \frac{2r}{K} x.$$  \(13\)

Evaluation at \(x^*\) results in

$$\dot{\eta} = f'(x^*) \eta = \left(r - \frac{2r}{K} x^*\right) \bigg|_{x=x^*} \eta.$$  \(14\)

In the case \(x_1^* = 0\) we have

$$\dot{\eta} = \left(r - \frac{2r}{K} x\right) \bigg|_{x=x^*=0} \eta = r \eta,$$

and in the case \(x_2^* = K\) we have

$$\dot{\eta} = \left(r - \frac{2r}{K} x\right) \bigg|_{x=x^*=K} \eta = (r - 2r) \eta = -r \eta.$$  \(15\)

3.2 Linear stability analysis of fixed points

In order to evaluate the stability of the above fixed points we need to evaluate the sign of the slope \(f'(x^*)\).

- For \(x_1^* = 0\) we get: \(f'(x^*) = \left(r - \frac{2r}{K} x\right) \bigg|_{x=x^*=0} = r\). The fixed point is **unstable** (the sign is positive).
- For \(x_2^* = K\) we get: \(f'(x^*) = \left(r - \frac{2r}{K} x\right) \bigg|_{x=x^*=K} = -r\). The fixed point is **stable** (the sign is negative).

The positive and negative slopes of the linearisations about the fixed points are shown in Fig.4.

**Figure:** Algebraic decay near a fixed point: Family of numerical solutions of \(\dot{x} = -x^3\). The initial condition of the solution shown with the red colour is \(x(0) = 1\). An algebraic decay path \(x(t) \sim t^{-\delta}\), where \(\delta\) is constant, is shown with the blue colour.
4 Existence and uniqueness of solutions of 1-D systems

Existence and uniqueness: Solutions to $\dot{x} = f(x)$ exist and they are unique if $f(x)$ and $f'(x)$ are continuous, i.e, the function $f$ is continuously differentiable.

Graphically speaking uniqueness implies that the graphs of solutions $x(t)$ for different initial conditions $x(0)$ do not cross each other, cf. Slide 3. A single initial condition $x(0)$ can not result in more than one solution (more than one future both in forward or backward time $t$).

5 Impossibility of oscillations in 1-D systems

The 1-D systems are limited in what they can describe. The possible behaviours of solution $x(t)$ as $t \to \infty$, for $\dot{x} = f(x)$ are:

(i) $x(t) \to \pm \infty$ as $t \to \infty$.
(ii) $x(t) \to x^*$ as $t \to \infty$.

The behaviours that are not possible include: oscillations, periodic solutions, chaos (defined/explained in future lectures). Two examples of impossible behaviours are shown in Fig. 5

Figure 5: Examples of impossible behaviours in 1-D systems: (Left) oscillation with attenuation, (Right) harmonic oscillation.
Why is this true? Let’s study Fig. 6 where the flow is always damped out at the stable fixed points for \( t \to \infty \). By definition 1-D systems do not have inertial terms (\( \ddot{x} \)) this means that all forces are always balanced out by the damping (\( \dot{x} \)), see Eq. (1).

![Phase portrait of an arbitrary 1-D flow problem](image)

Figure 6: Phase portrait of an arbitrary 1-D flow problem where \( x_0 = x(0) \) is the initial condition.

## 6 Bifurcation

Bifurcation: The term is related to models with instabilities, sudden changes and transitions.

With the change of a parameter the qualitative structure of the vector field may change dramatically — fixed points may be created or destroyed, or they might change their stability. Such a change is called the bifurcation.

**Bifurcation point** is the value of the parameter at which the change (bifurcation) occurs.

In this section our goal is to familiarise ourselves with all possible bifurcation dynamics that can occur in 1-D systems.

### 6.1 Saddle-node bifurcation

![Phase portraits of Eq. (17) undergoing saddle-node bifurcation](image)

Figure 7: Phase portraits of Eq. (17) undergoing saddle-node bifurcation.

Basic mechanism for creation or destruction of fixed points. The **standard** example is

\[
\dot{x} = r + x^2,
\]

(17)
where $r$ is the **control parameter**. Figure 7 shows phase portraits corresponding to different values of $r$. Fixed points exist only for $r \leq 0$ and they are given by

$$\dot{x} = 0 \Rightarrow r + x^2 = 0 \Rightarrow x^* = \pm i\sqrt{r}, \quad r < 0,$$

(18)

where $i$ is the imaginary unit. The **bifurcation point** value $r = 0$ and it occurs at spatial point $x = 0$, i.e., $(x, r) = (0, 0)$. This example also shows the origin of the mysterious **half-stable fixed point** present in Eqs. (7) and (8). In those examples we simply caught the system at the moment of bifurcation. An identical general behaviour can also be show to happen for

$$\dot{x} = r - x^2.$$

(19)

### 6.1.1 Bifurcation diagram

**Bifurcation diagram** is obtained by plotting the graph of fixed point $x^*$ as a function of $r$, analytically expressed by (18). The bifurcation diagram of Eq. (17) is shown in Fig. 8.

![Bifurcation diagram](image)

Figure 8: Bifurcation diagram of the saddle-node bifurcation. The unstable branch is shown with the dashed line and the stable branch with the continuous bold line.

The following numerical file shows the quantitatively accurate phase portraits and bifurcation diagrams discussed in this section.

**Numerics: cdf#1, nb#1**

Bifurcation diagrams in 1-D systems: saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation (supercritical), pitchfork bifurcation (subcritical).

### 6.1.2 Not so clear example of a saddle-node bifurcation

Let’s consider the following problem

$$\dot{x} = r + x - \ln(1 + x),$$

(20)

where $r$ is the control parameter. It is impossible to determine the dynamics of the fixed points algebraically (Try it, and/or see the numerical file linked below).

**Numerics: cdf#1, nb#1**

Bifurcation diagrams in 1-D systems: saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation (supercritical), pitchfork bifurcation (subcritical).

**Note:** The red error messages from Mathematica are left in intentionally. The symbolic calculation packages (CAS – computer algebra system) can not handle the problem using inverse functions.

We employ the graphical approach to find the fixed points and their dependents on control parameter $r$. This means that we need to solve

$$\dot{x} = 0 \Rightarrow r + x - \ln(1 + x) = 0,$$

(21)

or equivalently

$$r + x = \ln(1 + x),$$

(22)
for variable $x$. We can use an arbitrary axis $y$ to plot the left hand side and the right hand side of Eq. (22)

$$y = r + x \quad \text{and} \quad y = \ln(1 + x).$$

(23)

Figure 9 shows the behaviour of curves (23) as the value of $r$ is changed. The saddle-node bifurcation occurs when tangent intersection of these curves takes place. We write and it must hold

$$\begin{cases} r + x = \ln(1 + x), \\ \frac{d}{dx}(r + x) = \frac{d}{dx}\ln(1 + x). \end{cases}$$

(24)

As stated above the first equation in Sys. (24) is hard to solve algebraically, but the second one is easier. After taking the derivatives we get

$$1 = \frac{1}{1 + x} \quad \Rightarrow \quad x^* = 0.$$ 

(25)

The corresponding $r$ value can be found from the first equation in Sys. (24)

$$(r + x)|_{x^*=0} = \ln(1 + x)|_{x^*=0},$$

(26)

$$r = \ln 1 = 0.$$ 

(27)

Thus, the bifurcation point $(x, r) = (0, 0)$.

**NUMERICS: CDF#1, NB#1**

**Bifurcation diagrams in 1-D systems:** saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation (supercritical), pitchfork bifurcation (subcritical).

Figure 10: Bifurcation diagram of Eq. (20). The stable branch is shown with the solid line and the unstable branch is shown with the dashed line.

File content: A quantitatively precise behaviour of the dynamics shown in Fig. 9; the phase portrait and bifurcation diagram corresponding to Eq. (20). Figure 10 shows the resulting diagram.
### 6.1.3 Normal form

Let’s take a closer look at the previous result. The series expansion of Eq. (20) is in the form

\[
\dot{x} = r + x - \ln(1 + x) \approx \left[ \text{Maclaurin series expansion of } \ln(1 + x) \right]_0 \approx r + x - \left( x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \right) \approx r + \frac{x^2}{2} + O(x^3). \tag{28}
\]

This result is similar to our first example given by Eq. (17). Surely, after neglecting \(O(x^3)\) terms, the qualitative dynamics of this system will be exactly the same. Many systems can be reduced to the form (17) near their respective fixed points. We say that the **normal form** of the saddle-node bifurcation is

\[
x = r \pm x^2. \tag{29}
\]

If you can simplify or reduce a system to this form then that system will have all the properties corresponding to the normal form.

### 6.2 Transcritical bifurcation

![Phase portraits of Eq. (31) for different values of \(r\).](image)

**Figure 11:** Phase portraits of Eq. (31) for different values of \(r\).

![Bifurcation diagram of Eq. (31).](image)

**Figure 12:** Bifurcation diagram of Eq. (31). The stable branches are shown with the bold solid lines and the unstable branches are shown with the dashed lines.

The normal form is given by

\[
\dot{x} = rx \pm x^2, \tag{30}
\]

where \(r\) is the control parameter. Let’s study the case

\[
\dot{x} = rx - x^2 = x(r - x). \tag{31}
\]

Fixed points for this system are

\[
\begin{align*}
x^*_1 &= 0, \quad \forall r, \\
x^*_2 &= r.
\end{align*}
\]

Figures 11 and 12 show the phase portraits for the varied values of \(r\) and the corresponding bifurcation diagram, respectively. **Note:** \(x^*_1 = 0\) can’t be destroyed but its stability can be changed. From Fig. 12 it is clear that the bifurcation point \((x, r) = (0, 0)\). These result can be compared against the result found using computer.
Bifurcation diagrams in 1-D systems: saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation (supercritical), pitchfork bifurcation (subcritical).

--- Skip if needed: start ---

For training purposes let’s analyse the stability of the bifurcation diagram branches algebraically. In order to determine the stability of the fixed points one needs to analyse the slopes of phase portrait curves at said fixed points $x^*$ (positive slope corresponds to unstable point and negative one to stable point).

\[
f'(x) = \frac{d}{dx}(rx - x^2) = r - 2x
\]

(33)

In case of $x_1^* = 0$

\[
f'(x_1^*) = f'(0) = r, \quad x_1^* \text{ is } \begin{cases} \text{stable for } r < 0, \\ \text{unstable for } r > 0. \end{cases}
\]

(34)

In case of $x_2^* = r$

\[
f'(x_2^*) = f'(r) = -r, \quad x_2^* \text{ is } \begin{cases} \text{stable for } r > 0, \\ \text{unstable for } r < 0. \end{cases}
\]

(35)

Indeed, these result agrees with the graphical results presented above.

--- Skip if needed: stop ---

6.3 Pitchfork bifurcation

Pitchfork bifurcation occurs in systems with symmetry. Normal form is given by

\[
\dot{x} = rx \pm x^3,
\]

(36)

where $r$ is the control parameter.

6.3.1 Supercritical pitchfork bifurcation

Figure 13: Phase portrait of Eq. (37) for different values of $r$. In the case $r = 0$ solutions near the fixed point $x^* = 0$ decay algebraically.

Figure 14: Bifurcation diagram. The stable branches are shown with the solid bold lines and the unstable branch is shown with dashed line.

Let’s study the case

\[
\dot{x} = rx - x^3.
\]

(37)
Fixed points for this system are
\[
\begin{align*}
x_1^* &= 0, \quad \forall r, \\
x_2^* &= \pm \sqrt{r}, \quad r \geq 0.
\end{align*}
\] (38)

Figures 13 and 14 show the phase portraits for varied values of \( r \) and the corresponding bifurcation diagram, respectively. From Fig. 14 it is clear that the bifurcation point \((x, r) = (0, 0)\). These result can be compared against the result found using computer.

**Numerics:** CDF#1, NB#1

Bifurcation diagrams in 1-D systems: saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation (supercritical), pitchfork bifurcation (subcritical).

![Phase portraits and bifurcation diagrams](image)

Figure 15: Phase portrait of Eq. (39) for different values of \( r \). In the case \( r = 0 \) the solutions near the fixed point \( x^* = 0 \) decay algebraically.

![Bifurcation diagram](image)

Figure 16: Bifurcation diagram. The stable branch is shown with the solid bold line and the unstable branches are shown with the dashed lines.

### 6.3.2 Subcritical pitchfork bifurcation

Let’s consider the other case of the normal form (36) where
\[
\dot{x} = rx + x^3.
\] (39)

Fixed points for this system are the same as for the previous case but with different stability types. The fixed point are
\[
\begin{align*}
x_1^* &= 0, \quad \forall r, \\
x_2^* &= \pm i\sqrt{r}, \quad r < 0,
\end{align*}
\] (40)

where \( i \) is the imaginary unit. Figures 15 and 16 show the phase portraits for varied values of \( r \) and the corresponding bifurcation diagram, respectively. From Fig. 16 it is clear that the bifurcation point \((x, r) = (0, 0)\). These result can be compared against the result found using computer.

**Numerics:** CDF#1, NB#1

Bifurcation diagrams in 1-D systems: saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation (supercritical), pitchfork bifurcation (subcritical).
### Revision questions

1. What does linearisation of a nonlinear system imply?
2. Linearise the following 1-D system

   \[ \dot{x} = x^3 - x \]  

   (41)

3. What is bifurcation?
4. What is bifurcation diagram?
5. Are oscillation possible in 1-D systems?
6. Why are oscillations impossible in 1-D systems?
7. What does uniqueness of solutions imply in the context of phase space trajectories?