Lecture 4: 2-D homogeneous linear systems, classification of fixed points in 2-D systems, Lyapunov stability, basin of attraction

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Handout: Classification of fixed points of linear homogeneous 2-D systems
1 Linear homogeneous 2-D systems

1.1 Introduction

We continue our discussion of second-order systems in the form

\[ \dot{x} = f(x), \]

where \( f \) is an arbitrary function, \( x = (x, y)^T \) and \( x \in \mathbb{R}^2 \). The component form of Eq. (1) is

\[
\begin{cases}
\dot{x} = f(x, y), \\
\dot{y} = g(x, y),
\end{cases}
\]

where \( f \) and \( g \) are functions. The fixed point in the case (1) is defined as follows

\[ \dot{x} = 0 \implies f(x) = 0, \]

and for the component form (2)

\[
\begin{cases}
\dot{x} = 0 \implies f(x) = 0, \\
\dot{y} = 0 \implies g(x) = 0.
\end{cases}
\]

In this lecture we will discuss second-order linear homogeneous systems with constant coefficients. The following slide show an example of a homogeneous differential equation and compares it to its nonhomogeneous counterpart.

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**Second-order differential equations**

- Homogeneous differential equation:
  \[ ax + bx + cx = 0 \iff \begin{cases} \dot{x} = y \\ ay = -by + cx \end{cases} \]

- Nonhomogeneous differential equation:
  \[ ax + bx + cx = f(x) \iff \begin{cases} \dot{x} = y \\ ay = -by + cx + f(x) \end{cases} \]

- Non-autonomous differential equation:
  \[ ax + bx + cx = f(t) \iff \begin{cases} \dot{x} = y \\ ay = -by + cx + f(t) \end{cases} \]

In all of the above cases \( a, b, \) and \( c \) are the constant coefficients, and \( f \) is an arbitrary (linear) function.

In a more general setting \( a, b, \) and \( c \) may depend on time \( t \).

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Also included here is the non-autonomous system mentioned in Lecture 1.

1.2 2-D phase portrait plotting

As introduced in the previous lecture, phase portrait of second-order Sys. (2) is constructed by plotting vectorfield \( \dot{x} = (x, y)^T = (f(x, y), g(x, y))^T \) against the independent variables \( x \) and \( y \) in \( xy \)-plane, as shown in Fig. 1 and on Slide 4. A trajectory of vector field represents a single solution of Sys. (2) corresponding to an initial condition—a single point in the plane \( (x_0, y_0) \), where \( x_0 = x(0), y_0 = y(0) \).

**Essence of the course (Henri Poincaré):** Construction of phase portrait allows us to find all qualitatively different trajectories (solutions) of a system without solving the system itself explicitly (analytically).
Figure 1: (Left) Phase portrait of a 2-D or a second-order problem. A trajectory is shown with the continuous line where \( t_1 < t_2 < t_3 < t_4 \). (Right) Idea behind the construction of a 2-D phase portrait (entire vector field for the shown ranges of \( x \) and \( y \)) using computer. The vector field vectors are shown for points \((x, y)\) placed in a uniform grid that is shown with the dashed lines.

An example of phase portrait plotting procedure and different visualisation styles of phase portraits are shown in the following numerical file.

Vector field plotting. Integrated solution and phase portrait of damped harmonic oscillator.

1.3 Why bother with linear homogeneous 2-D systems?

In nonlinear systems it is harder to determine the stability of fixed points, due to a possibility of more complex dynamics present in the phase portraits. For this reason, in future lectures, we will be linearising nonlinear systems and then studying the stability and dynamics of the corresponding linearised systems with the aim to gain insight into the original nonlinear systems. The conditions under which this approach is allowed and feasible will be discussed in the lectures to come. This means that we need to familiarise ourselves with linear second-order systems and their dynamics.

The general form of linear homogeneous Sys. (2) is the following:

\[
\begin{align*}
\dot{x} &= ax + by, \\
\dot{y} &= cx + dy,
\end{align*}
\]

where \(a, b, c, d \in \mathbb{R} \) are the constant coefficients. By defining \( \vec{x} = (x, y)^T \) we rewrite Sys. (5) in the matrix form

\[
\dot{\vec{x}} = A\vec{x} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{x},
\]

where matrix \( A \) is the system or coefficient matrix of Sys. (5). Dynamics of linear systems is fully determined by the eigenvalues \( \lambda \) and eigenvectors \( \vec{v} \) of matrix \( A \).
1.4 Examples

1.4.1 Harmonic oscillator

Harmonic oscillator is described by a linear system in the following form:

\[
\begin{align*}
\ddot{x} + \ddot{x} = 0 & \Rightarrow \begin{cases} 
\dot{x} = y, \\
\dot{y} = -x.
\end{cases} \quad (7)
\end{align*}
\]

where \(x\) is the displacement. The matrix form of this system for \(\vec{x} = (x, y)^T\) and appropriately selected system matrix \(A\) is

\[
\vec{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x}.
\]

The solution and phase portrait of harmonic oscillation (7) are shown in the following numerical file.

Vector field plotting. Integrated solution and phase portrait of damped harmonic oscillator.

1.4.2 Damped harmonic oscillator

Damped harmonic oscillator is described by a linear system in the following form:

\[
\begin{align*}
\ddot{x} + \ddot{x} + \frac{\ddot{x}}{\text{Damping}} = 0 & \Rightarrow \begin{cases} 
\dot{x} = y, \\
\dot{y} = -x - y.
\end{cases} \quad (9)
\end{align*}
\]

where \(x\) is the displacement and the last term on the right-hand side of the equation is the damping or friction term. The matrix form of this system for \(\vec{x} = (x, y)^T\) and appropriately selected system matrix \(A\) is

\[
\vec{x} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \vec{x}.
\]

The solution and phase portrait of damped harmonic oscillation (9) are shown in the following numerical file.

Vector field plotting. Integrated solution and phase portrait of damped harmonic oscillator.

2 Theory of 2-D linear systems

As stated above, fixed point dynamics of a linear 2-D system is determined by the eigenanalysis of its system matrix \(A\). The following is a short reminder of your linear algebra courses.

Let’s consider 2-D linear system given in the form

\[
\vec{x} = A\vec{x},
\]

where

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

\(\{a, b, c, d\} \in \mathbb{R}\), and \(\vec{x} = (x, y)^T\). The fixed point \(\vec{\pi} = 0\) and the phase portrait near \(\vec{\pi} = 0\) is fully determined by the eigenvalues and eigenvectors of system matrix \(A\).

What are the eigenvalues and vectors of a system? Following is a short reminder of your linear algebra courses.

We seek a straight line solution (assumption) in the following form:

\[
\vec{x}(t) = \vec{v}\lambda^t,
\]

where \(\vec{v}\) is the eigenvector and \(\lambda\) is the eigenvalue. Eqs. (7) and (9) yield

\[
\vec{x} = \vec{v}\lambda^t \Leftrightarrow A\vec{x} = A(\vec{v}\lambda^t),
\]

\[
\vec{v}\lambda^t = A\vec{v}\lambda^t \Leftrightarrow \vec{v}\lambda^t = \lambda\vec{v},
\]

a useful relationship between the eigenvalues and eigenvectors of a system.
3 Classification of fixed points in 2-D linear systems

All possible behaviours found in linear homogeneous second-order systems are presented here. The classification of fixed points is presented in terms of system matrix determinant Δ and trace τ.

3.1 CASE 1: Saddle node (also called, saddle point or saddle)

Criterion for determining the fixed point:

\[ Δ < 0. \]

(11)

This criterion holds for real and distinct (not equal) eigenvalues and linearly independent (not on the same line) eigenvectors. Since \( Δ = λ_1 λ_2 \), we must have \( λ_1 \leq 0 \) and \( λ_2 \geq 0 \). The determination of saddle node fixed point does not depend on trace τ, \((-∞ < τ < ∞)\). Figures 2 and 3 show the dynamics of a saddle.

Stability: The fixed point \( \vec{x}^* = \vec{0} = (0, 0)^T \) is always unstable, although the phase portrait has one stable eigendirection (other one is unstable).

![Figure 2: Phase portrait of saddle node fixed point, where eigenvalues \( λ_1 > 0 \) and \( λ_2 < 0 \).](image)

General solution of the system:

\[ \vec{x}(t) = C_1 e^{λ_1 t} \vec{v}_1 + C_2 e^{λ_2 t} \vec{v}_2, \]

(12)

where \( C_i \) are the integration constants and \( \vec{v}_i \) are the eigenvectors. See Figs. 2 and 3.
Calculation of matrix determinant, trace, eigenvalues and eigenvectors using computer. Classification of fixed points in 2-D linear homogeneous systems.

An example of quantitatively accurate phase portrait of the dynamics shown in Fig. 2.

Figure 3: Phase portrait and eigenvectors. Eigenvectors are shown with the red arrows. (Left) Saddle node. (Right) Saddle node, different flow directions.

3.2 CASE 2a: Node

Criterion for determining the fixed point:

\[
\Delta > 0 \quad \text{and} \quad \tau^2 - 4\Delta > 0.
\] (13)

This criterion holds for real and distinct eigenvalues having the same sign, \( \lambda_1, \lambda_2 \geq 0 \). Eigenvectors are linearly independent. The criterion \( \tau^2 - 4\Delta > 0 \) simply means that we are located outside the region defined by \( \tau^2 - 4\Delta = 0 \) shown in the overview plot in Fig. 17 and Slide 10. Figures 4 and 5 show the dynamics of a node.

Figure 4: Phase portrait of stable node where \( \lambda_1 < \lambda_2 < 0 \). As \( t \to \infty \) trajectories approach the fixed point tangentially to the slower eigendirection.

Stability: The fixed point \( \bar{x}^* = \bar{0} \) may be either stable (attracting sink) or unstable (repelling source). If

\[
\tau < 0,
\] (14)

then we have a stable node, and from (13) it also follows that \( \lambda_1, \lambda_2 < 0 \). If

\[
\tau > 0,
\] (15)

then we have an unstable node, and from (13) it follows that \( \lambda_1, \lambda_2 > 0 \).

General solution of the system:

\[
\bar{x}(t) = C_1 e^{\lambda_1 t} \bar{v}_1 + C_2 e^{\lambda_2 t} \bar{v}_2,
\] (16)

where \( C_i \) are the integration constants and \( \bar{v}_i \) are the eigenvectors. See Figs. 4 and 5.
Calculation of matrix determinant, trace, eigenvalues and eigenvectors using computer. Classification of fixed points in 2-D linear homogeneous systems.

An example of quantitatively accurate phase portrait of the dynamics shown in Fig. 4.

![Phase portrait](image)

Figure 5: Phase portrait and eigenvectors. Eigenvectors are shown with the red arrows. (Left) Stable node. (Right) Unstable node.

### 3.3 CASE 2b: Spiral

**Criterion** for determining the fixed point:

\[
\Delta > 0 \quad \text{and} \quad \tau^2 - 4\Delta < 0. \tag{17}
\]

This criterion holds for complex and distinct eigenvalues. The criterion \(\tau^2 - 4\Delta < 0\) simply means that we are located inside the region defined by \(\tau^2 - 4\Delta = 0\) shown in the overview plot in Fig. 17 and Slide 10. Figures 6 and 7 show the dynamics of a spiral.

![Phase portrait](image)

Figure 6: Phase portrait of stable spiral in the case \(\mu < 0\).

Physical interpretation of complex eigenvalues \(\lambda = \mu + i\omega\) is the following: real part \(\mu\) is the decay rate (\(\mu < 0\)) of the spiral, and imaginary part \(\omega\) is the rotation rate.

**Stability:** The fixed point \(\bar{x}^* = 0\) may be either stable (attracting sink) or unstable (repelling source). If

\[
\tau < 0, \tag{18}
\]

which also implies that \(\mu < 0\), then we have a stable spiral. If

\[
\tau > 0, \tag{19}
\]

which also implies that \(\mu > 0\), then we have an unstable spiral.

**General solution** of the system: A linear combination of

\[
e^{\mu_1 t} \cos \omega_1 t, \tag{20}
\]

and

\[
e^{\mu_2 t} \sin \omega_2 t. \tag{21}
\]
Calculation of matrix determinant, trace, eigenvalues and eigenvectors using computer. Classification of fixed points in 2-D linear homogeneous systems.

An example of quantitatively accurate phase portrait of the dynamics shown in Fig. 6.

Figure 7: Phase portrait. (Left) Stable spiral. (Right) Unstable spiral.

Which way is vector field rotating? This question is not an issue when one uses computer to construct the phase portrait. But, let’s say you need to sketch the portrait by hand. System matrix $A$ does not explicitly give you the direction. In order to determine the direction simply calculate one vector and the direction of the entire flow becomes clear.

Figure 8: Determination of the rotation direction. Flow vector, calculated for coordinate $(x, y)^T = (1, 0)^T$, is shown with the bold arrow. Phase portrait features Lyapunov stable fixed point at its origin.

Example: We consider a simple case that was introduced above—harmonic oscillator defined by Eq. (7)

$$
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x.
\end{align*}
$$

Let’s say we are interested in a vector located at $(x, y)^T = (1, 0)^T$ (shown with the red bullet in Fig. 8). The resulting vector is

$$
\begin{align*}
\dot{x} &= y = 0 \\
\dot{y} &= -x = -1 \\
\Rightarrow \quad \dot{\mathbf{x}} &= \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.
\end{align*}
$$

This vector is shown on the phase portrait in Fig. 8. From here it is clear that the field is rotating in a clockwise direction. 2-D systems feature a new type of stability called the Lyapunov stability. A fixed point is said to be Lyapunov stable if a solution starts near a fixed point and it stays near it for $t \to \infty$ (note: a more rigorous definition exists). The fixed point of harmonic oscillator shown in Fig. 8 is Lyapunov stable.

3.4 CASE 3: Center

Criterion for determining the fixed point:

$$
\Delta > 0 \quad \text{and} \quad \tau = 0.
$$
This criterion holds for pure imaginary eigenvalues \( \lambda = \pm \mu \) (complex conjugate pair). Figures 9 and 10 show the dynamics of a center.

![Phase portrait of Lyapunov stable center.](image)

**Figure 9:** Phase portrait of Lyapunov stable center.

**Stability:** The fixed point \( \vec{x}^* = \vec{0} \) is Lyapunov stable. This is true because the real part, associated with the decay rate of the rotation, of the pure imaginary eigenvalues \( \mu = 0 \).

**General solution** of the system: A linear combination of

\[
\cos \omega_1 t, \quad (24)
\]

and

\[
\sin \omega_2 t. \quad (25)
\]

**Numerics:** cdf#2, nb#2

Calculation of matrix determinant, trace, eigenvalues and eigenvectors using computer. Classification of fixed points in 2-D linear homogeneous systems.

An example of quantitatively accurate phase portrait of the dynamics shown in Fig. 9.

![Phase portrait. (Left) Center featuring clockwise rotation. (Right) Center featuring counterclockwise rotation.](image)

**Figure 10:** Phase portrait. (Left) Center featuring clockwise rotation. (Right) Center featuring counterclockwise rotation.

### 3.5 CASE 4a: Degenerate node (of the first type)

**Criterion** for determining the fixed point:

\[
\Delta > 0 \quad \text{and} \quad \tau^2 - 4\Delta = 0. \quad (26)
\]

This criterion holds for real and repeated (equal) eigenvalues and for one uniquely determined eigenvector (the other can be anything). The criterion \( \tau^2 - 4\Delta = 0 \) simply means that we are located on the line defined by \( \tau^2 - 4\Delta = 0 \) shown in the overview plot in Fig. 17 and Slide 10. Figures 11 and 12 show the dynamics of this fixed point.

**Stability:** The fixed point \( \vec{x}^* = \vec{0} \) can be either stable or unstable. If

\[
\tau < 0, \quad (27)
\]

which also implies \( \lambda_1 = \lambda_2 < 0 \), then the fixed point is stable. If

\[
\tau > 0, \quad (28)
\]

which implies \( \lambda_1 = \lambda_2 > 0 \), then the fixed point is unstable.

**General solution:** Omitted from this document.
Calculation of matrix determinant, trace, eigenvalues and eigenvectors using computer. Classification of fixed points in 2-D linear homogeneous systems.

An example of quantitatively accurate phase portrait of the dynamics shown in Fig. 11.

3.6 CASE 4b: Degenerate node (of the second type: Star)

**Criterion** for determining the fixed point:

\[ \Delta > 0 \quad \text{and} \quad \tau^2 - 4\Delta = 0. \]  

(29)

This criterion holds for real and repeated (equal) eigenvalues and for no uniquely determined eigenvectors (both can be anything, every direction is an eigendirection). The criterion \( \tau^2 - 4\Delta = 0 \) simply means that we are located on the line defined by \( \tau^2 - 4\Delta = 0 \) shown in the overview plot in Fig. 17 and Slide 10. Figures 13 and 14 show the dynamics of this fixed point.

**Stability:** The fixed point \( \bar{x}^* = \bar{0} \) can be either stable or unstable. If

\[ \tau < 0, \]  

(30)

which also implies \( \lambda_1 = \lambda_2 < 0 \), then the fixed point is stable. If

\[ \tau > 0, \]  

(31)

which implies \( \lambda_1 = \lambda_2 > 0 \), then the fixed point is unstable.

**General solution:** Omitted from this document.
3.7 CASE 5a: Non-isolated fixed point, a line of fixed points

Criterion for determining the fixed point:

\[
\Delta = 0 \quad \text{and} \quad \tau \neq 0.
\]  

(32)

This criterion holds for real eigenvalues. On the overview plot, shown in Fig. 17 and Slide 10, we are located on the vertical \( \tau \)-axis. Figures 15 and 16 show the dynamics of non-isolated fixed point.

Stability: The fixed points (line) can be either Lyapunov stable or unstable. If

\[
\tau < 0,
\]

then the fixed point is Lyapunov stable. If

\[
\tau > 0,
\]

(34)

then the fixed point is unstable.

General solution of the system:

\[
\ddot{x}(t) = C_1 e^{\lambda_1 t} \vec{v}_1 + C_2 e^{\lambda_2 t} \vec{v}_2,
\]

(35)

where \( C_1 \) are the integration constants and \( \vec{v}_i \) are the eigenvectors. See Figs. 15 and 16.
3.8 CASE 5b: Non-isolated fixed point, a plane of fixed points

Criterion for determining the fixed point:

\[ \Delta = 0 \quad \text{and} \quad \tau = 0. \]  \hspace{1cm} (36)

All points on the plane are fixed points. Nothing happens and nothing can happen! On the overview plot, shown in Fig. 17 and on Slide 10, criterion (36) is located at the origin.

3.9 Summary overview

A nice and concise way of summarising the classification discussed in this lecture is shown on Slide 10 and in Fig. 17.
Figure 17: Classification of fixed points of linear homogeneous 2-D systems, where trace $\tau = \lambda_1 + \lambda_2$ and determinant $\Delta = \lambda_1 \lambda_2$ are determined by $2 \times 2$ system matrix $A$.

The presented classification can also be summarised by the following flowchart:

- if $\Delta < 0$:
  Isolated fixed point
  CASE 1: **Saddle node**

- if $\Delta = 0$:
  Non-isolated fixed points
  - if $\tau < 0$:
    CASE 5a: **Line of Lyapunov stable fixed points**
  - if $\tau = 0$:
    CASE 5b: **Plane of fixed points**
  - if $\tau > 0$:
    CASE 5a: **Line of unstable fixed points**

- if $\Delta > 0$:
  Isolated fixed point
  - if $\tau < -\sqrt{4\Delta}$:
    CASE 2a: **Stable node**
  - if $\tau = -\sqrt{4\Delta}$:
    - if there is one uniquely determined eigenvector (the other is non-unique):
CASE 4a: **Stable degenerate node**
○ if there are no uniquely determined eigenvectors (both are non-unique):
  CASE 4b: **Stable star**
• if $-\sqrt{4\Delta} < \tau < 0$:
  CASE 2b: **Stable spiral**
• if $\tau = 0$:
  CASE 3: **Center**
• if $0 < \tau < \sqrt{4\Delta}$:
  CASE 2b: **Unstable spiral**
• if $\tau = \sqrt{4\Delta}$:
  ○ if there is one uniquely determined eigenvector (the other is non-unique):
    CASE 4a: **Unstable degenerate node**
  ○ if there are no uniquely determined eigenvectors (both are non-unique):
    CASE 4b: **Unstable star**
• if $\sqrt{4\Delta} < \tau$:
  CASE 2a: **Unstable node**

### 3.10 A note on degenerate nodes

![Image](image_url)

Figure 18: Comparison of critically damped system solution time-series shown on the left to damped system solution time-series shown on the right.

The degenerate nodes rarely occur in engineering applications. They represent critically damped or over-damped systems. These second-order systems achieve equilibrium without being allowed to oscillate. Figure 18 compares a possible solution of a critically damped oscillator to a solution of damped oscillator.

### 4 Basin of attraction

The definition of the basin of attraction is given on Slide 13. The notion of basin of attraction will be used/expanded in future lectures.
Revision questions
1. How to plot 2-D phase portrait of a system?
2. What are 2-D homogeneous linear systems?
3. What are non-homogeneous systems?
4. Classification of fixed points in 2-D systems.
5. Sketch a saddle node fixed point.
6. Sketch a stable node fixed point.
7. Sketch an unstable node fixed point.
8. Sketch a stable spiral (fixed point).
9. Sketch an unstable spiral (fixed point).
10. Sketch a center (fixed point).
11. Sketch a stable non-isolated fixed point.
12. Sketch an unstable non-isolated fixed point.
13. 2-D homogeneous nonlinear systems.
14. What does it mean that a fixed point is Lyapunov stable?
15. Give an example of Lyapunov stable fixed point.