LECTURE 6: 2-D CONSERVATIVE SYSTEMS AND CENTERS, CLOSED ORBITS AND LIMIT-CYCLES, NULL-CLINE, HETEROCLINIC ORBIT, DULAC’S CRITERION, POINCARÉ-BENDIXSON THEOREM

Contents

1 Properties of conservative systems ................................................................. 2
   1.1 Nonlinear centers in 2-D conservative systems ............................................. 2
   1.2 Example: Mathematical pendulum ............................................................. 2
       1.2.1 Model and its fixed points ............................................................... 2
       1.2.2 Proof of conserved quantity ........................................................... 3
       1.2.3 Linear analysis .................................................................................. 4

2 Limit-cycles ........................................................................................................... 5

3 Testing for closed orbits ........................................................................................ 6
   3.1 Dulac’s criterion ........................................................................................... 6
       3.1.1 Example 1 ............................................................................................ 6
       3.1.2 Example 2 (home assignment) ............................................................... 7
   3.2 Proof of Dulac’s criterion ............................................................................... 8
   3.3 Poincaré-Bendixson theorem ......................................................................... 8
       3.3.1 Example 1 ............................................................................................ 9
       3.3.2 Example 2: Glycolysis ........................................................................ 11

Material teaching aids: Phase portrait of mathematical pendulum, periodic phase portrait of mathematical pendulum
1 Properties of conservative systems

1.1 Nonlinear centers in 2-D conservative systems

Figure 1: **Isolated fixed point** where $\epsilon \ll 1$. No other fixed points exist in close vicinity to the fixed point, shown with the dashed circle.

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**Theorem:** Suppose $\dot{x} = f(x)$ is conservative and $f$ is continuously differentiable in $x \in \mathbb{R}^2$. $E(x)$ is a conserved quantity and $\bar{x}^*$ is an isolated fixed point. If that fixed point is a local minimum or maximum of $E(x)$, then that isolated fixed point $\bar{x}^*$ is a **center**, i.e., all trajectories close to $\bar{x}^*$ are closed orbits.

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Figure 2: Closed trajectories close to the local minimum or maximum of the conserved quantity $E(x)$. Closed orbits are shown with the red lines.

1.2 Example: Mathematical pendulum

1.2.1 Model and its fixed points
Mathematical pendulum

Mathematical pendulum\(^1\) is given in the form

\[
\ddot{\theta} + \sin \theta = 0, \tag{1}
\]

where \(\theta\) is the angular displacement. For angular velocity \(\omega = \dot{\theta}\) we write

\[
\begin{aligned}
\dot{\theta} &= \omega, \\
\dot{\omega} &= -\sin \theta. 
\end{aligned} \tag{2}
\]

\(^1\)See Mathematica .nb file uploaded to course webpage.

Normalised and dimensionless model of mathematical pendulum is given by

\[
\ddot{\theta} + \sin \theta = 0, \tag{1}
\]

where \(\theta\) is the angular displacement shown in Fig. 3. For angular velocity \(\omega = \dot{\theta}\) we rewrite the Eq. (1) as follows

\[
\begin{aligned}
\dot{\theta} &= \omega, \\
\dot{\omega} &= -\sin \theta. 
\end{aligned} \tag{2}
\]

Figure 3: Mathematical pendulum where \(\theta\) is the downwards angular displacement.

Notice that Eq. (1) is explicitly independent of both \(\dot{\theta}\) and \(t\); hence there is no damping or friction of any kind, and there are no time-dependent external driving forces. This indicates that the system is conservative. Let’s study the dynamics of the pendulum model (1) for \(-2\pi \leq \theta \leq 2\pi\). Fixed points \((\theta^*, \omega^*)\) of the system are the following:

\[
\ldots, (-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0), \ldots. \tag{3}
\]

1.2.2 Proof of conserved quantity

We use Eq. (1) to prove the existence of conserved quantity, which in this case is energy (see Lecture 5)

\[
\ddot{\theta} + \sin \theta = 0 \quad | \cdot \dot{\theta},
\]

equivalently and for \(\omega = \dot{\theta}\) we write

\[
\frac{d}{dt} \left( \frac{\omega^2}{2} - \cos \theta \right) = 0, \tag{5}
\]

where the first term corresponds to the kinetic energy and the second to the potential energy. The potential energy \(V\) follows directly from Eq. (1). Since force \(F\) is defined as follows

\[
F = -\frac{dV}{d\theta}, \tag{6}
\]
and Eq. (1) is a balance of forces we derive the potential

\[ V = -\int (-\sin \theta) d\theta = \int \sin \theta d\theta = -\cos \theta + C, \]  

(7)

where the integration constant \( C = 0 \). From result (5) it is clear that the sum of kinetic and potential energies is indeed conserved in time. The conserved energy or more specifically the Hamiltonian has the form

\[ E(\theta, \omega) = \frac{\omega^2}{2} - \cos \theta = \text{const}. \]  

(8)

Slide 5 shows the energy values as plotted against angle \( \theta \) and angular velocity \( \omega \). The following numerical file contains the code that produced the plot.

**NUMERICS: cdf#1, nb#1**

Mathematical pendulum and heteroclinic orbit. Integrated numerical solution.

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### 1.2.3 Linear analysis

Jacobian matrix of Sys. (2) is

\[ J = \begin{pmatrix} \frac{\partial \dot{\theta}}{\partial \theta} & \frac{\partial \dot{\theta}}{\partial \omega} \\ \frac{\partial \dot{\omega}}{\partial \theta} & \frac{\partial \dot{\omega}}{\partial \omega} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\cos \theta & 0 \end{pmatrix}. \]  

(9)

Evaluation of the matrix about the system fixed points \((\theta^*, \omega^*) = (0, 0), (\pm \pi, 0), (\pm 2\pi, 0)\) is the following:

\[ J\big|_{(\theta^*, \omega^*)} = J\big|_{(0,0)} = J\big|_{(\pm 2\pi,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]  

(10)

here the trace \( \tau = 0 \) and the determinant \( \Delta = 1 \). According to the linear fixed point classification we have a **linear center** and according to the theorem presented in Sec. 1.1, this center is also a true **nonlinear center** because it is located at a local minimum of the Hamiltonian (conserved quantity) (8), see Slide 5;

\[ J\big|_{(\theta^*, \omega^*)} = J\big|_{(\pm \pi,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]  

(11)
here the matrix determinant $\Delta = -1$ and we have a saddle. Now we have all the information required to sketch the phase portrait. Slide 6 shows the phase portrait. The qualitatively accurate phase portrait and the interactive numerical solution of Eq. (1) or Sys. (2) are presented in the following numerical file.

![Mathematical pendulum and heteroclinic orbit. Integrated numerical solution.](image1)

We have encountered a new type of dynamics represented by the heteroclinic orbit. One could also argue that this orbit is homoclinic since the pendulum is $2\pi$-periodic. Fixed points $(\pm 2\pi, 0)$ and $(0, 0)$ or $(-\pi, 0)$ and $(\pi, 0)$ are actually the same fixed points.

## 2 Limit-cycles

A limit-cycle is an isolated closed trajectory. Isolated means that neighbouring trajectories are not closed; they spiral either toward or away from the limit-cycle, see Fig. 4.

![Limit-cycles](image2)

Figure 4: Limit-cycles are shown with the closed continuous or dashed red lines. Continuous lines indicate the stable and dashed the unstable limit-cycles. (Left) Stable or attracting limit-cycle. (Middle) Unstable or repelling limit-cycle. (Right) Half-stable limit-cycle that is stable from the inside and unstable from the outside.

If all neighbouring trajectories approach a limit-cycle, we say that this limit-cycle is stable or attracting. Otherwise the limit-cycle is unstable or repelling, or in exceptional cases, half-stable. Half-stable limit-cycle can be stable from inside and unstable from outside as shown in Fig. 4 (Right) of vice versa.

Stable limit-cycles are very important scientifically—they model systems that exhibit self-sustained oscillations. In other words, these systems oscillate even in the absence of external periodic forcing. Of the countless examples that could be given, we mention only a few: the beating of a heart; the periodic firing of a pacemaker neuron; daily rhythms in human body temperature and hormone secretion; chemical
reactions that oscillate spontaneously; and dangerous self-excited vibrations in bridges and airplane wings that can lead to structure damage and failure. In each case, there is a standard/nominal oscillation of some preferred period, waveform, and amplitude. If the system is perturbed slightly, it always returns to the nominal cycle. **Limit-cycles are inherently nonlinear phenomena** they can not occur in linear systems (by linear system we mean $\dot{x} = Ax$, where system matrix $A$ has constant and real valued coefficients).

### 3 Testing for closed orbits

#### 3.1 Dulac’s criterion

Dulac’s criterion is a negative criterion used to rule out limit-cycles.

**Dulac’s criterion**

Let $\dot{x} = \vec{f}(x)$ be a continuously differentiable vector field defined on a **simply connected** subset $R$ of a plane. If there exists a continuously differentiable, real valued function $g(x)$ such that

$$\text{div}(g\vec{x}) = \nabla \cdot (g\vec{x}),$$

has one sign throughout $R$, then there are no closed orbits lying entirely in $R$.  

In a **simply connected** region it is possible to shrink the circumference or perimeter of the region to be infinitely small (not a technical definition). Figure 5 shows a comparison between a simply connected region and not simply connected region.

![Figure 5](image.png)

Figure 5: (Left) Simply connected region $R$ having no holes. (Right) Not simply connected region.

#### 3.1.1 Example 1

Show that system

$$\begin{cases} \dot{x} = x(2 - x - y), \\ \dot{y} = y(4x - x^2 - 3), \end{cases}$$

has no closed orbits in simply connected region $R$, where $x > 0$ and $y > 0$, shown in Fig. 6.

We have to come up with function $g(x)$. Let’s pick (educated guess)

$$g = \frac{1}{xy},$$
Now we need to study the sign of
\[
\text{div}(g\tilde{x}) = \nabla \cdot (g\tilde{x}) = \left( \frac{\partial}{\partial x} g\tilde{x} \right) + \left( \frac{\partial}{\partial y} g\tilde{x} \right)
\]
\[
= x(2 - x - y) + y(4x - x^2 - 3) = -\frac{1}{y} < 0, \quad (14)
\]
in region \( R \) because \( y > 0 \). Since the sign is strictly negative we do not have closed orbits in the region \( R \). We can confirm this conclusion using a computer. Numerical file linked below shows the phase portrait of Sys. (12).

### NUMERICS: cdf#2, nb#2

Dulac’s criterion and limit-cycles. A numerical example: integrated solution and phase portrait.

Phase portrait of Sys. (12), shown below, confirms that there are no closed orbits in region \( R \).

![Phase portrait](image)

Figure 7: Phase portrait of Sys. (12) featuring three unstable fixed points shown with the empty bullets.

#### 3.1.2 Example 2 (home assignment)

Show that system
\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x - y + x^2 + y^2,
\end{align*}
\]
has no closed orbits in simply connected region \( R \subseteq \mathbb{R}^2 \).

We have to come up with function \( g(x) \). Let’s pick (educated guess)
\[
g = e^{-2x}. \quad (16)
\]

Now we need to study the sign of
\[
\text{div}(g\tilde{x}) = \nabla \cdot (g\tilde{x}) = \frac{\partial}{\partial x} (e^{-2x}y) + \frac{\partial}{\partial y} [e^{-2x}(-x - y + x^2 + y^2)] = -2e^{-2x}y - e^{-2x} + 2e^{-2x}y = -e^{-2x} < 0, \quad (17)
\]
in region \( R \). Since the sign is strictly negative we do not have closed orbits in the region \( R \subseteq \mathbb{R}^2 \).
Dulac’s criterion and limit-cycles. Numerical example: integrated solution and phase portrait.

Phase portrait of Sys. (15), shown below, confirms that there are no closed orbits in region \( R \in \mathbb{R}^2 \).

Figure 8: Phase portrait of Sys. (15) featuring a stable spiral and the an unstable saddle node.

### 3.2 Proof of Dulac’s criterion

#### Proof by contradiction, Dulac’s criterion

Let \( C \) be a closed orbit in subset \( R \), and let \( A \) be the region inside \( C \).

Green’s theorem:

\[
\iint_A (\nabla \cdot \vec{F}) \, dA = \oint_C (\vec{F} \cdot \hat{n}) \, dl \tag{6}
\]

If \( \vec{F} = g\hat{x} \), then

\[
\iint_A \left[ \nabla \cdot (g\hat{x}) \right] \, dA = \oint_C \left( g\hat{x} \cdot \hat{n} \right) \, dl \tag{7}
\]

Therefore there is no closed orbit \( C \) in \( R \).

Reminder: The dot product of orthogonal vectors is 0.

### 3.3 Poincaré-Bendixson theorem

Now that we know how to rule out closed orbits, we turn to the opposite task: finding a method to establish that closed orbits exist in particular systems. The following theorem is one of the few results in this direction.
Suppose that:
- $R$ is a closed, bounded subset in $\mathbb{R}^2$, called the trapping region;
- $\dot{\mathbf{x}} = f(\mathbf{x})$ is a continuously differentiable vector field on an open set containing $R$;
- $R$ does not contain any fixed points ($P$); and
- there exists a trajectory $C$ that is “confined” in $R$, in the sense that it starts in $R$ and stays in $R$ for all future time.

Then either $C$ is a closed orbit, or it spirals toward a closed orbit as $t \to \infty$. In either case, $R$ contains a closed orbit (shown as a heavy curve in the above figure).

Trapping region $R$ usually has an annular shape. Figure 9 shows a relatively simple trapping region. Trapping regions are not simply connected subsets of phase plane.

Figure 9: Trapping region $R$ with annular donut-like shape.

**Practical tip for constructing a trapping region:** Find an annulus such that vector field points into it on its boundaries. According to Poincaré-Bendixson theorem a closed orbit will be trapped in that annulus. Also, find and prefer annuli that have minimal area. Figure 10 shows an example of such annular region.

Figure 10: Annulus $R$ with vector field vectors pointing into it on its boundaries.

### 3.3.1 Example 1

Consider the following system given in polar coordinates

$$
\begin{align*}
\dot{r} &= r(1 - r^2) + \mu r \cos \theta, \\
\dot{\theta} &= 1,
\end{align*}
$$

(18)

where $\mu$ is the control parameter. Show that closed orbits exist for small positive $\mu$, i.e., $0 < \mu \ll 1$.

If $\mu = 0$ then the system takes the form

$$
\begin{align*}
\dot{r} &= r(1 - r^2), \\
\dot{\theta} &= 1.
\end{align*}
$$

(19)

This system is decoupled. Angular velocity $\dot{\theta}$ is constant and positive. The behaviour of a trajectory in the radial direction is described by the first equation of this system. This equation is similar to previously
introduced Logistic equation. 1-D phase portrait of the first equation is shown in Fig. 11. We can sketch the phase portrait corresponding to Sys. (19) by combining the above observations. Figure 12 shows the resulting phase portrait and the stable limit-cycle associated with the carrying capacity \( r^* = K = 1 \) of the first equation present in Sys. (19).

![Figure 11: Phase portrait of 1-D equation featured in (19) where the quantity similar the carrying capacity \( K \) of the Logistic equation is \( r^* = K = 1 \).](image1)

![Figure 12: Phase portrait shown in polar coordinates corresponding to Sys. (19). Stable limit-cycle is shown with the red closed trajectory.](image2)

Let’s consider the full system where \( \mu \neq 0 \). According to Poincaré-Bendixson theorem we need to construct an annular trapping region. Figure 13 shows a promising candidate. We seek two concentric circles with radii \( r = r_{\text{min}} \) and \( r = r_{\text{max}} \) such that \( \dot{r} < 0 \) on the outer circle and \( \dot{r} > 0 \) on the inner circle. Then the annulus \( 0 < r_{\text{min}} \leq r \leq r_{\text{max}} \) will be our trapping region. Note that there are no fixed points in the annulus since \( \dot{\theta} > 0 \neq 0 \); hence if \( r_{\text{min}} \) and \( r_{\text{max}} \) can be found, then Poincaré-Bendixson theorem will imply the existence of a closed orbit.

![Figure 13: Annular trapping region shown in polar coordinates. Radial distance \( r = r_{\text{min}} \) is the inner boundary and \( r = r_{\text{max}} \) is the outer boundary of the proposed annulus.](image3)

Let’s show that vector field defined by (18) flows into the selected annulus on its boundaries. Firstly, we consider the inner boundary \( r = r_{\text{min}} \) where it must hold that \( \dot{r} > 0 \) for all \( \theta \)

\[
\dot{r} = r(1 - r^2) + \mu r \cos \theta = r(1 - r^2 + \mu \cos \theta) > 0.
\]

Since \( -1 \leq \cos \theta \leq 1 \), a sufficient condition for \( r_{\text{min}} \) is

\[
r(1 - r^2 - \mu) > 0 \quad \div r,
\]

\[
1 - r^2 - \mu > 0, \quad \text{and} \quad r^2 < 1 - \mu,
\]
any such \( r \) will do as long as \( \mu < 1 \), which is fine since in our case \( \mu \ll 1 \).

Secondly, we consider the outer boundary \( r = r_{\text{max}} \) where it must hold that \( \dot{r} < 0 \) for all \( \theta \)

\[
\dot{r} = r(1 - r^2 + \mu \cos \theta) < 0.
\] (25)

Since \(-1 \leq \cos \theta \leq 1\), a sufficient condition for \( r_{\text{max}} \) is

\[
r(1 - r^2 + \mu) < 0 \quad \mid \div r, 
\] (26)

\[
1 - r^2 + \mu < 0, 
\] (27)

\[
r^2 > 1 + \mu, 
\] (28)

\[
r > \sqrt{1 + \mu}. 
\] (29)

Now that \( r_{\text{min}} \) and \( r_{\text{max}} \) have been found we have proven the existence of a limit-cycle. Obtained result can be confirmed using a computer.

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### 3.3.2 Example 2: Glycolysis

This example is less contrived compared to the previous one. Let’s consider a simplified dimensionless model of glycolysis given by

\[
\begin{align*}
\dot{x} &= -x + ay + x^2y, \\
\dot{y} &= b - ay - x^2y,
\end{align*}
\] (30)

where \(a\) and \(b\) are the kinetic parameters, \(x\) and \(y\) are the concentrations of ADP (adenosine diphosphate) and F6P (fructose-6-phosphate) molecules, respectively. Here, \(x, y, a, b > 0\). Using Poincaré-Bendixon theorem show that chemical oscillations are possible. Determine the values of \(a\) and \(b\) that lead to oscillating reaction.

A useful tools for studying phase portraits are **null-clines**. The null-clines are curves on the 2-D phase portrait corresponding to \( \dot{x} = 0 \) and \( \dot{y} = 0 \). In the case \( \dot{x} = 0 \) and for the first equation in Sys. (30) we write

\[
\dot{x} = 0 \quad \Rightarrow \quad -x + ay + x^2y = 0 \quad \Rightarrow \quad y_{\dot{x}=0}(x) = \frac{x}{a + x^2}. 
\] (31)
In the case $\dot{y} = 0$ and for the second equation in Sys. (30) we write

$$\dot{y} = 0 \quad \Rightarrow \quad b - ay - x^2 y = 0 \quad \Rightarrow \quad y_{\dot{y}=0}(x) = \frac{b}{a + x^2}. \quad (32)$$

The point where null-clines intersect each other corresponds to the fixed point. Null-clines (31) and (32) are sketched by hand (explained during the lecture) in Fig. 15, along with some representative vectors. An easy way to determine the flow direction of the vector field defined by Sys. (30) is to estimate or calculate one vector and deduce the others by relying on the fact that the field must be continuous. We estimate that for $x \gg 1$ vector components $\dot{x} \approx x^2 y > 0$ and $\dot{y} \approx -x^2 y < 0$. This vector, shown in the upper-right corner of Fig. 15, is used as a starting point to populate the phase portrait with other vectors.

![Figure 15: Null-clines (31) and (32). Flow direction of the vector field is shown with the arrows. Fixed point shown with the hollow bullet is assumed to be unstable.](image)

Slide 14 shows a promising trapping region. If we can show that vector field defined by Sys. (30) flows into the trapping region, then we have proven the existence of a closed orbit.

![Slide 14: Glycolysis, trapping region](image)

Null-clines (31) and (32) are shown with the continuous black lines.

First, we focus on the outer boundary. It is evident that the flow direction on outer boundaries shown with the continuous red lines on Slide 14 is indeed pointing into the annulus. But, it is not so clear with the upper-right slanted part of the boundary shown with the dashed line. How was point $x = b$ selected and why the slope $dy/dx = -1$ was selected?

As shown above the slope of the field vectors for $x \gg 1$ can be easily approximated from the algebraically dominant parts of the field (30) $\dot{x} \approx x^2 y$ and $\dot{y} \approx -x^2 y$. The slope $dy/dx = \dot{y}/\dot{x} \approx -1$. This result is not
accurate enough. We should compare the sizes of \( \dot{x} \) and \( -\dot{y} \) more precisely

\[
\dot{x} = (-\dot{y}), \\
-x + ay' + x^2y' + (b - ay' - x^2y'), \\
b - x \Rightarrow x = b.
\]

Hence

\[
-\dot{y} > \dot{x}, \quad \text{if } x > b.
\]

This inequality implies that the vector field points inward on the slanted line shown on Slide 14, because \( dy/dx \) is more negative than \(-1\), and therefore the vectors are steeper than the slanted line. Also, the question regarding the selection of point \( x = b \) is answered here.

Now let’s focus on the vector field flow through the inner boundary of the trapping region.

In order to ensure repelling unstable fixed points for \( \Delta > 0 \) trace \( \tau \) has to be positive. The dividing line between repelling unstable fixed points and stable ones is \( \tau = 0 \). Solving

\[
\tau = 0 \Rightarrow \frac{2b^3}{a + b^2} - 1 - a - b^2 = 0
\]

for \( b \) gives

\[
b(a) = \sqrt{\frac{2}{a} (1 - 2a \pm \sqrt{1 - 8a})}. \tag{15}
\]

This result defines a line in the parameter space of Sys. (9). For parameters \( a \) and \( b \) in the region corresponding to \( \tau > 0 \), we are guaranteed that Sys. (9) has a closed orbit—oscillating reaction.
Figure 16: $\Delta$ vs. $\tau$ fixed point classification graph. The marked region is populated by repellers: unstable spirals, unstable centers, unstable stars, and unstable degenerate nodes.

The result shown on Slide 19 is calculated using the following numerical file.

**Numerics: cdf#5, nb#5**

Glycolysis phase portrait and null-clines. Numerical solution of glycolysis model.

We conclude that the annulus shown on Slide 14 is indeed the trapping region and a closed orbit exists in it. Numerical integration shows that closed orbit is a stable limit-cycle. Slides 20 and 21 show the phase portrait and numerically integrated time-domain results for a typical case where $a = 0.07$, $b = 0.53$.

**Slides: 20, 21**

Numerical file that was used to generate the above figures is shown below.

**Numerics: cdf#5, nb#5**

Glycolysis phase portrait and null-clines. Numerical solution of glycolysis model.

Phase portrait and null-clines for a typical case where $a = 0.07$, $b = 0.53$. 
Poincaré-Bendixson theorem is one of the central results of nonlinear dynamics. It says that the dynamical possibilities in the phase plane are very limited: if a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory must eventually approach a closed orbit. Nothing more complicated is possible. This result depends crucially on the two-dimensionality of the plane. In higher-dimensional systems \( n \geq 3 \), Poincaré-Bendixson theorem no longer applies. The theorem also implies that chaos can never occur in the phase plane.

**Reading suggestion**

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**Revision questions**

1. Expand on the connection between 2-D conservative systems and centers.
2. Sketch a heteroclinic orbit.
3. What is limit-cycle?
4. Sketch a stable limit-cycle.
5. Sketch an unstable limit-cycle.
6. Sketch a half-stable (stable from outside) limit-cycle.
7. Sketch a half-stable (stable from inside) limit-cycle.
8. Define and sketch a null-cline.
9. What is Dulac’s criterion?
10. State Poincaré-Bendixson theorem.
11. Does Poincaré-Bendixson theorem apply to 3-D systems?
12. Can chaos occur in 2-D systems?