Lecture №10: 1-D unimodal maps, Lorenz, Logistic and sine maps, period doubling bifurcation, tangent bifurcation, intermittency, orbit diagram, Feigenbaum constants

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Lecture outline

- Lorenz map and unstable limit-cycles, graphical approach
- Connection between 3-D chaotic systems and 1-D maps
- Period-p points (period-p orbit)
- Logistic map
- Analysis and properties of Logistic map
- Sine map
- Period doubling bifurcation in unimodal maps
- Tangent bifurcation in unimodal maps
- Orbit diagram (or Feigenbaum diagram) or fig tree diagram
- Feigenbaum diagram
- Universal aspect of period doubling in unimodal maps
- Universal route to chaos
- Feigenbaum constants $\delta$ and $\alpha$
Logistic map\(^1\) has the form

\[
x_{n+1} = rx_n(1 - x_n), \quad x_0 \in [0, 1], \quad r \in [0, 4], \quad n \in \mathbb{Z}^+,
\]

where \(r\) is the control parameter.


\(^1\)See Mathematica .nb file (cobweb diagram and orbit diagram) uploaded to the course webpage.
Lyapunov exponent of Logistic map

Chaos is characterised by sensitive dependence on initial conditions. If we take two close-by initial conditions, say $x_0$ and $y_0 = x_0 + \eta$ with $\eta \ll 1$, and iterate them under the map, then the difference between the two time series $\eta_n = y_n - x_n$ should grow exponentially

$$|\eta_n| \sim |\eta_0 e^{\lambda n}|,$$

(2)

where $\lambda$ is the Lyapunov exponent. For maps, this definition leads to a very simple way of measuring Lyapunov exponents. Solving (2) for $\lambda$ gives

$$\lambda = \frac{1}{n} \ln \left| \frac{\eta_n}{\eta_0} \right|.$$

(3)

By definition $\eta_n = f^n(x_0 + \eta_0) - f^n(x_0)$. Thus

$$\lambda = \frac{1}{n} \ln \left| \frac{f^n(x_0 + \eta_0) - f^n(x_0)}{\eta_0} \right|.$$

(4)
Lyapunov exponent of Logistic map

For small values of $\eta_0$, the quantity inside the absolute value signs is just the derivative of $f^n$ with respect to $x$ evaluated at $x = x_0$:

$$\lambda = \frac{1}{n} \ln \left| \frac{df^n}{dx} \right|_{x=x_0}.$$  \hspace{1cm} (5)

Since $f^n(x) = f(f(f(\ldots f(x)))) \ldots$, by the chain rule

$$\left| \frac{df^n}{dx} \right|_{x=x_0} = \left| f'(f^{n-1}(x_0)) \cdot f'(f^{n-2}(x_0)) \cdot \ldots \cdot f'(x_0) \right|$$

$$= \left| f'(x_{n-1}) \cdot f'(x_{n-2}) \cdot \ldots \cdot f'(x_0) \right| = \left| \prod_{i=0}^{n-1} f'(x_i) \right|.$$ \hspace{1cm} (6)

Our expression for the Lyapunov exponent becomes

$$\lambda = \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right| = \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|.$$ \hspace{1cm} (7)
Lyapunov exponent of Logistic map

\[ \lambda = \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right| = \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|. \]

Lyapunov exponent is the large iterate \( n \) limit of this expression, and so we have,

\[ \lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|. \] (8)

This formula can be used to study Lyapunov exponent\(^2\) as a function of control parameter \( r \)

\[ \lambda(r) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i, r)|. \] (9)

\(^2\)See Mathematica .nb file uploaded to course webpage.
Logistic map, period-2 window

Period-2 window for $3 \leq r < 1 + \sqrt{6}$.

Figure: Logistic map where $r = 3.18$ and $x_0 = 0.35$. Fixed points (f.p.s) in the case where $r < 3$ are shown with the grey bullets. Period-2 points of $f(x)$ map for $r \geq 3$ are shown with the blue bullets. The fixed points of $f^2(x)$ map for $r \geq 3$ are shown with the red bullets.
Logistic map, period-2 window

Period-2 window for $3 \leq r < 1 + \sqrt{6}$.

\[
\begin{align*}
  f(p) &= rp(1 - p) = q, \\
  f(q) &= rq(1 - q) = p,
\end{align*}
\]

(10)

where period-2 point values $p$ and $q$ are the f.p.s of $f(x)$ map.

On the other hand it also holds

\[
\begin{align*}
  f(p) &= f(f(q)) \equiv f^2(q) = r[rq(1 - q)][1 - (rq(1 - q))] = q, \\
  f(q) &= f(f(p)) \equiv f^2(p) = r[rp(1 - p)][1 - (rp(1 - p))] = p,
\end{align*}
\]

(11)

\[
\Rightarrow f^2(x) = r[rx(1 - x)][1 - (rx(1 - x))] = x,
\]

(12)

where period-2 point values $p$ and $q$ are the f.p.s of $f^2(x)$ map.
Stability of f.p.s of $f^2$ map in period-2 orbit

We need to know the slopes of period-2 points

$$\begin{cases} f(p) = rp(1 - p) = q, \\ f(q) = rq(1 - q) = p. \end{cases}$$

According to the chain rule it holds that

$$(f^2(x))' \equiv (f(f(x)))' = f'(f(x)) \cdot f'(x). \quad (13)$$

In our case

$$\begin{cases} (f^2(p))' = f'(f(p)) \cdot f'(p) = f'(q) \cdot f'(p) \\ (f^2(q))' = f'(f(q)) \cdot f'(q) = f'(p) \cdot f'(q) \end{cases} \Rightarrow (f^2(p))' = (f^2(q))'. \quad (14)$$

Above follows from the commutative property of multiplication.
Logistic map, period doubling

Even number periods.

\( r_n \) – bifurcation point, onset of stable period-\( 2^n \) orbit.

\[
\begin{align*}
    r_1 &= 3.0 & \text{period-2} \\
    r_2 &= 1 + \sqrt{6} \approx 3.44949 & \text{period-4} \\
    r_3 &\approx 3.54409 & \text{period-8} \\
    r_4 &\approx 3.56441 & \text{period-16} \\
    r_5 &\approx 3.56875 & \text{period-32} \\
    r_6 &\approx 3.56969 & \text{period-64} \\
    \vdots & & \vdots \\
    r_\infty &\approx 3.569946 & \text{period-}\infty
\end{align*}
\]

\( r_\infty \) – onset of chaos (the accumulation point).

\[
\delta = \lim_{n \to \infty} \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} \approx 4.669201609...
\quad (15)
\]
Orbit diagram and period doubling
Zooming into the Logistic map, self-similarity

The logistic map

\[ x \rightarrow a \times (1 - x) \]

Credit: https://www.youtube.com/user/logicedges/
Intermittency\(^3\) and period-3 window

Transient chaos and intermittency in dynamical systems. Tangent bifurcation occurs at \( r = 1 + \sqrt{8} \approx 3.8284 \) (period-3 orbit).

Iterates of Logistic map shown for \( r = 3.8282 \) and \( x_0 = 0.15 \).

\(^3\)See Mathematica .nb file uploaded to course webpage.
1-D sine map\(^4\). The sine map has the form

\[ x_{n+1} = r \sin(\pi x_n), \quad x_0 \in [0, 1], \quad r \in [0, 1], \quad n \in \mathbb{Z}^+, \tag{16} \]

where \( r \) is the control parameter.


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\(^4\)See Mathematica .nb file (cobweb diagram and orbit diagram) uploaded to the course webpage.
1-D unimodal maps and Feigenbaum constants

\[ \delta = \lim_{n \to \infty} \frac{\Delta_{n-1}}{\Delta_n} = \lim_{n \to \infty} \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} \approx 4.669201609\ldots \]  
\[ \alpha = \lim_{n \to \infty} \frac{d_{n-1}}{d_n} \approx -2.502907875\ldots \]
Conclusions

- Lorenz map and unstable limit-cycles, graphical approach
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Revision questions

- What is cobweb diagram?
- What is recurrence map or recurrence relation?
- What is 1-D map?
- How to find fixed points of 1-D maps?
- What is Lorenz map?
- What is Logistic map?
- What is sine map?
- What is period doubling?
- What is period doubling bifurcation?
- What is tangent bifurcation.
- Do odd number periods (period-p orbits) exist in chaotic systems?
- Do even number periods (period-p orbits) exist in chaotic systems?
Revision questions

- Can maps produce transient chaos?
- Can maps produce intermittency?
- Can maps produce intermittent chaos?
- What is orbit diagram (or Feigenbaum diagram)?
- What are Feigenbaum constants?