Normal Lax Double Functors Preserve Companions

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Abstract

In this paper we review the definitions of weak double categories, lax double functors between them, and connection structures within them; we survey several illustrative examples of each concept, prove the title proposition, and consider some of its applications. Throughout, we use the graphical calculus of dual pasting diagrams to help clarify the 2-dimensional structures we encounter.

1 Introduction

The proximate purpose of this article is to prove the title proposition, along with its three duals. The result itself is probably folklore, at the least there are closely related results to be found in the literature on double categories, which are surveyed below. Still, I have found this result in particular to be useful, and I believe that it is worth recording a direct, algebraic proof of it.

A more general motivation is to demonstrate a style of presentation based on the graphical calculus of dual pasting diagrams (or "string diagrams") that is useful for perspicuously recording and reasoning about many sorts of compositional structure, including that of double categories.

Double categories, along with their dimensional generalizations, are useful tools for reasoning about compositional systems in which there are multiple interacting notions of morphism. From one perspective, cubical n-categories are just another member of the menagerie of shapes of higher-dimensional categorical structure, along with globular, simplicial, opetopic, and other even more fantastic beasts. From this perspective, the various forms of higher-dimensional categories should be somehow equivalent, at least in the dimensional colimit, and the main feature differentiating them is the combinatorics that relates their structure at different dimensions.

However, from another perspective, the cubical shape affords a tractable way to represent, not only structure *at* different dimensions (e.g. a 2-dimensional

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*This research was supported by the ESF funded Estonian IT Academy research measure (project 2014-2020.4.05.19-0001).

square vs a 1-dimensional edge), but also structure *in* different dimensions[1] (e.g. the two distinct 1-dimensional subspaces of a square), along with relations that may hold between them (e.g. a square that is symmetric about its diagonal has the same structure in both dimensions).

2 Double Categories

Strict double categories were introduced by Ehresmann in [Ehr62]. The weak or pseudo- form considered here has been studied extensively by Grandis and Paré in a sequence of articles beginning with [GP99].

Intuitively, a double category is made up of 0-dimensional objects, two distinct sorts of 1-dimensional morphisms, and 2-dimensional squares whose opposite faces are morphisms of the same sort and adjacent faces are morphisms of opposite sorts. Squares compose in both dimensions by pasting along a shared boundary morphism. This operation is well-defined in the sense that any way of pasting together a compatible diagram yields the same composite [DP93]. A concise way to make this idea precise is to say that a double category is a pseudo category internal to the 2-category of (suitably small) categories, which amounts to the following.

Definition 2.1 (double category)
A (pseudo) *double category* $\mathbb{D}$, consists of two ordinary categories, $\mathbb{D}_0$ and $\mathbb{D}_1$, related by four functors:

$$\mathrm{L}, \mathrm{R} : \mathbb{D}_1 \to \mathbb{D}_0 \quad , \quad \mathrm{U} : \mathbb{D}_0 \to \mathbb{D}_1 \quad , \quad - \odot - : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$$

where the pullback is taken over $(\mathrm{R} \, , \mathrm{L})$, such that

identity boundaries: $\mathrm{U} \cdot \mathrm{L} = \mathrm{id}\, \mathbb{D}_0 = \mathrm{U} \cdot \mathrm{R}$

composition boundaries: $- \odot - \cdot \mathrm{L} = \pi_0 \cdot \mathrm{L}$ and $- \odot - \cdot \mathrm{R} = \pi_1 \cdot \mathrm{R}$

together with coherent natural isomorphisms with the following components

unitors: $\lambda(\mathrm{M}) : \mathrm{U}(\mathrm{LM}) \odot \mathrm{M} \to \mathrm{M}$ and $\rho(\mathrm{M}) : \mathrm{M} \odot \mathrm{U}(\mathrm{RM}) \to \mathrm{M}$

associator: $\kappa(\mathrm{M} \, , \mathrm{N} \, , \mathrm{P}) : (\mathrm{M} \odot \mathrm{N}) \odot \mathrm{P} \to \mathrm{M} \odot (\mathrm{N} \odot \mathrm{P})$

We call objects of $\mathbb{D}_0$ *objects* of the double category $\mathbb{D}$, morphisms of $\mathbb{D}_0$ its *arrows*, objects of $\mathbb{D}_1$ its *proarrows*, and morphisms of $\mathbb{D}_1$ its *squares*.

The functors L and R pick out the "left" and "right" boundary objects of a proarrow, and arrows of a square, respectively. For a proarrow $\mathrm{M} : \mathbb{D}_1$, we use the barred arrow notation "$\mathrm{M} : \mathrm{A} \nrightarrow \mathrm{B}$" to indicate that $\mathrm{L}(\mathrm{M}) = \mathrm{A}$ and $\mathrm{R}(\mathrm{M}) = \mathrm{B}$. The functor U gives the *identity* proarrow on an object, and square on an arrow, and $- \odot -$ gives the *composite* of consecutive proarrows, and of squares in the proarrow dimension. For composition in the (strict) arrow dimension we use the notation "$- \cdot -$" with units "id". We write both sorts of composition in normal (in contrast to applicative) order.

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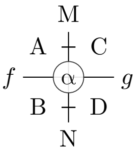
[1] the use of the term "dimension" for both concepts is perhaps unfortunate, but widespread, and context is usually sufficient to disambiguate them.

The coherence of the unitors and associator can be characterized in the same way as for bicategories, namely by the so-called "pentagon equation" relating terms of type $((\mathrm{L} \odot \mathrm{M}) \odot \mathrm{N}) \odot \mathrm{P} \to \mathrm{L} \odot (\mathrm{M} \odot (\mathrm{N} \odot \mathrm{P}))$ and the "triangle equation" relating those of type $(\mathrm{M} \odot \mathrm{U}) \odot \mathrm{N} \to \mathrm{M} \odot \mathrm{N}$. If the unitor natural isomorphisms are identities then the double category is called *unitary*. If the associator is an identity as well then it is *strict*. In the following we assume our double categories to be at least strict for identity proarrows, in the sense that $\mathrm{UA} \odot \mathrm{UA} = \mathrm{UA}$ and $\lambda(\mathrm{UA}) = \rho(\mathrm{UA}) = \kappa(\mathrm{UA}, \mathrm{UA}, \mathrm{UA}) = \mathrm{id}(\mathrm{UA})$. Such double categories are sometimes called *preunitary* [Gra19].

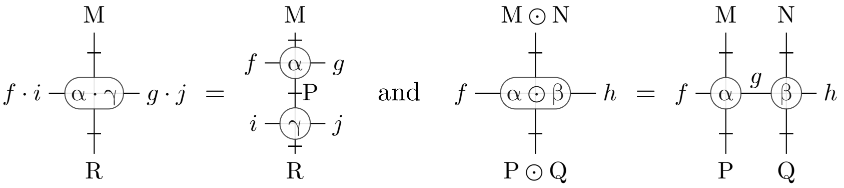
We write "${}^{\mathrm{M}}_{f}\Diamond^{g}_{\mathrm{N}}$" for the configuration of morphisms given by arrows $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{C} \to \mathrm{D}$ and proarrows $\mathrm{M} : \mathrm{A} \nrightarrow \mathrm{C}$ and $\mathrm{N} : \mathrm{B} \nrightarrow \mathrm{D}$. A square with this boundary, $\alpha : {}^{\mathrm{M}}_{f}\Diamond^{g}_{\mathrm{N}}$, can be depicted as either of the following dual diagrams, though we generally prefer the latter, and usually suppress object labels when they can be inferred or are irrelevant.

$$\begin{tikzcd}
\mathrm{A} \arrow[r, "\mathrm{M}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & \mathrm{C} \arrow[d, "g"] \\
\mathrm{B} \arrow[r, "\mathrm{N}"', "\shortmid"{marking}] & \mathrm{D}
\end{tikzcd}$$

,



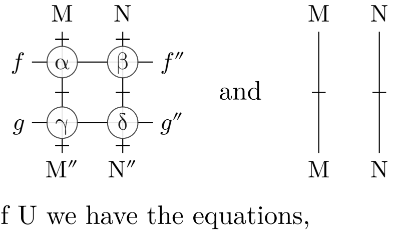
Composition of squares is depicted as pasting along the common boundary morphism in the appropriate dimension:



By the functoriality of $\odot$ we have the equations,

$$(\alpha \cdot \gamma) \odot (\beta \cdot \delta) = (\alpha \odot \beta) \cdot (\gamma \odot \delta) \quad \text{and} \quad \mathrm{id}\,\mathrm{M} \odot \mathrm{id}\,\mathrm{N} = \mathrm{id}(\mathrm{M} \odot \mathrm{N}),$$

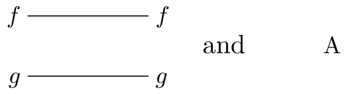
the former of which is a 2-dimensional associative law known as *middle-four exchange*. These imply that each of the following diagrams has a unique interpretation.



By the functoriality of U we have the equations,

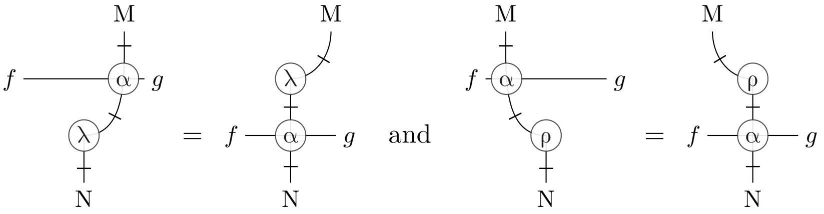
$$\mathrm{U}(f \cdot g) = \mathrm{U}f \cdot \mathrm{U}g \quad \text{and} \quad \mathrm{U}(\mathrm{id}\,\mathrm{A}) = \mathrm{id}(\mathrm{UA}),$$

the latter of which provides a well defined notion of (double) *identity square* on an object, which we write as "$\mathrm{id}^2\mathrm{A}$". These imply that each of the following diagrams has a unique interpretation.

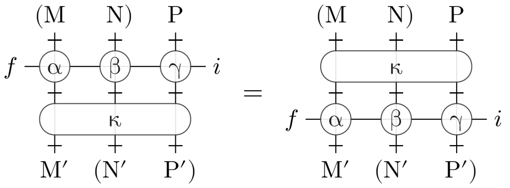


Note the convention of suppressing the drawing of composition unit cells. In order to declutter notation we may also use *dimensional promotion* to elide an "id" or "U" from a subterm when its dimension is evident from the context.

Unitor naturality implies that squares can "slide past" them, in the sense that for any $\alpha : {}^{\mathrm{M}}_{f}\Diamond^{g}_{\mathrm{N}}$ we have $(f \odot \alpha) \cdot \lambda\mathrm{N} = \lambda\mathrm{M} \cdot \alpha$ and $(\alpha \odot g) \cdot \rho\mathrm{N} = \rho\mathrm{M} \cdot \alpha$:



Similarly, squares can slide past an associator:



By convention we leave the associativity of a diagram implicit and rely on the reader to mentally insert reassociating isomorphisms as needed. For an introduction to string diagrams for double categories see Myers' [Mye16].

Analogous to the case for bicategories, there is a coherence result for double categories [GP99]. This is convenient because writing out all coherators explicitly can risk obscuring the main ideas of a construction. When presenting an equation that holds up to coherators, we will write "$\cong$" rather than "$=$" as a reminder that coherators may be (uniquely) inserted in order to unify the boundaries.

The objects of a double category together with either sort of morphism form a category (weak in the proarrow case unless the double category is strict), known as an *edge category*. Indeed, we can see not just 1-dimensional categories sitting inside of a double category, but globular 2-dimensional ones as well.

Definition 2.2 (globular square)
We call a square of a double category *globular* if it has identity boundary morphisms in (all but) one of its dimensions. A square with trivial proarrow boundary is called an *arrow disk* (shown on the left), and one with trivial arrow

boundary is called a *proarrow disk* (shown on the right):

$$\begin{tikzcd}
\mathrm{A} \arrow[r, "\mathrm{U}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & \mathrm{A} \arrow[d, "g"] \\
\mathrm{B} \arrow[r, "\mathrm{U}"', "\shortmid"{marking}] & \mathrm{B}
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
 & \mathrm{A} & \\
f \arrow[r, no head] & \boxed{\alpha} \arrow[r, no head] & g \\
 & \mathrm{B} &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{A} \arrow[r, "\mathrm{M}", "\shortmid"{marking}] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\beta"] & \mathrm{C} \arrow[d, "\mathrm{id}"] \\
\mathrm{A} \arrow[r, "\mathrm{N}"', "\shortmid"{marking}] & \mathrm{C}
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
 & \mathrm{M} \arrow[d, no head, "\shortmid"{marking}] & \\
\mathrm{A} & \boxed{\beta} \arrow[d, no head, "\shortmid"{marking}] & \mathrm{C} \\
 & \mathrm{N} &
\end{tikzcd}$$

The collections of arrow- and proarrow disks of a double category each determine a wide sub-double category. We can identify these sub-double categories of globular squares with actual 2-dimensional categories of globular shape, the *disk bicategories* of a double category, which is a strict 2-category in the arrow case. We overload the hom-notation for globular squares, writing "$f \twoheadrightarrow g$" and "$\mathrm{M} \to \mathrm{N}$" for ${}^{\mathrm{U}}_{f}\Diamond^{g}_{\mathrm{U}}$ and ${}^{\mathrm{M}}_{\mathrm{id}}\Diamond^{\mathrm{id}}_{\mathrm{N}}$, respectively[2]. These globular embeddings let us import constructions of formal category theory from bicategories [KS74; Lac09] into double categories.

Recall that a category is called *thin* if it has at most one morphism with a given boundary. By dimensional analogy, a double category is called *flat* if it has at most one square with a given boundary.

Here are some examples of double categories.

Example 2.1 (the double category of sets, functions and relations)
There is a double category SET, where the objects are sets, arrows are functions, and proarrows are relations. We express the proposition that relation $\mathrm{R} : \mathrm{A} \twoheadrightarrow \mathrm{B}$ relates elements $a \in \mathrm{A}$ and $b \in \mathrm{B}$ using the notation "$\mathrm{R}\,(a \dashrightarrow b)$". The double category SET is flat, with relational implications as squares:

$$\begin{tikzcd}
 & \mathrm{R} \arrow[d, no head, "\shortmid"{marking}] & \\
f \arrow[r, no head] & \boxed{\Downarrow} \arrow[r, no head] \arrow[d, no head, "\shortmid"{marking}] & g \\
 & \mathrm{V} &
\end{tikzcd}
\quad = \quad \forall\, a \in \mathrm{A}, b \in \mathrm{B} \;.\; \mathrm{R}\,(a \dashrightarrow b) \Rightarrow \mathrm{V}\,(fa \dashrightarrow gb)$$

with A, B to the upper left and right and X, Y to the lower left and right of the disk.

The composition of relations $\mathrm{R} : \mathrm{A} \twoheadrightarrow \mathrm{B}$ and $\mathrm{S} : \mathrm{B} \twoheadrightarrow \mathrm{C}$ is defined by the existential formula $\mathrm{R} \odot \mathrm{S}\,(a \dashrightarrow c) \coloneqq \exists\, b \in \mathrm{B} \;.\; \mathrm{R}\,(a \dashrightarrow b) \wedge \mathrm{S}\,(b \dashrightarrow c)$. Composition units are the homogeneous equality relations with $\mathrm{UA}\,(x \dashrightarrow y) \coloneqq x = y$. Relations are *extensional* in the sense that for $\mathrm{R}, \mathrm{R}' : \mathrm{A} \twoheadrightarrow \mathrm{B}$ we have $\mathrm{R} = \mathrm{R}'$ just in case $\forall\, a \in \mathrm{A}, b \in \mathrm{B} \;.\; \mathrm{R}\,(a \dashrightarrow b) \Leftrightarrow \mathrm{R}'\,(a \dashrightarrow b)$. By relation extensionality, SET is a strict double category.

Square composition in the arrow dimension is by the transitivity of implication. In the proarrow dimension, for any $a \in \mathrm{A}$ and $c \in \mathrm{C}$ in order to infer $\mathrm{V} \odot \mathrm{W}\,(fa \dashrightarrow hc)$ from $\mathrm{R} \odot \mathrm{S}\,(a \dashrightarrow c)$ we first apply the two implications being composed with the common element of B from the premise and then take the

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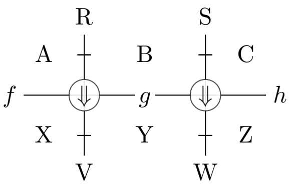
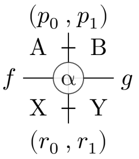
[2] Note that the latter is potentially ambiguous, since in the non-globular case the arrow boundaries of a square in $\mathbb{D}_1\,(\mathrm{M} \to \mathrm{N})$ need not be identities, so we must make clear when we mean a proarrow disk.

g-image of this element as our existential witness in the goal:



$$
\begin{aligned}
= \quad & \mathrm{R} \odot \mathrm{S}\,(a \dashrightarrow c) \\
:= \quad & \exists\, b \in \mathrm{B}\ .\ \mathrm{R}\,(a \dashrightarrow b) \wedge \mathrm{S}\,(b \dashrightarrow c) \\
\Rightarrow \quad & \exists\, b \in \mathrm{B}\ .\ \mathrm{V}\,(fa \dashrightarrow gb) \wedge \mathrm{W}\,(gb \dashrightarrow hc) \\
\Rightarrow \quad & \exists\, y \in \mathrm{Y}\ .\ \mathrm{V}\,(fa \dashrightarrow y) \wedge \mathrm{W}\,(y \dashrightarrow hc) \\
=: \quad & \mathrm{V} \odot \mathrm{W}\,(fa \dashrightarrow hc)
\end{aligned}
$$

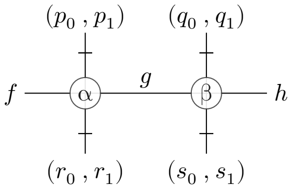
Example 2.2 (span double categories)
For a category $\mathbb{C}$ equipped with a pullback operation we define the *span double category* $\textsc{Span}\,\mathbb{C}$, where $(\textsc{Span}\,\mathbb{C})_0$ is $\mathbb{C}$ and a proarrow is a $\mathbb{C}$-span on the given boundary. A $(\textsc{Span}\,\mathbb{C})$-square is a $\mathbb{C}$-arrow between span apexes making the digram commute:



$$=\quad\begin{tikzcd}
& \mathrm{P} \arrow[dl, "p_0"'] \arrow[dr, "p_1"] \arrow[dd, "\alpha"] & \\
\mathrm{A} \arrow[dd, "f"'] & & \mathrm{B} \arrow[dd, "g"] \\
& \mathrm{R} \arrow[dl, "r_0"] \arrow[dr, "r_1"'] & \\
\mathrm{X} & & \mathrm{Y}
\end{tikzcd}$$

The composition of spans is given by composing the chosen $\mathbb{C}$-pullback span of their inner legs with their outer legs. Composition units are the identity spans $(\mathrm{id}\,,\mathrm{id})$. If the chosen pullbacks of cospans $(f\,,\mathrm{id})$ and $(\mathrm{id}\,,f)$ are the spans $(\mathrm{id}\,,f)$ and $(f\,,\mathrm{id})$, respectively then $\textsc{Span}\,\mathbb{C}$ is a unitary double category.

Square composition in the arrow dimension is by diagram pasting in $\mathbb{C}$. In the proarrow dimension it is given by the universal property of the pullback:

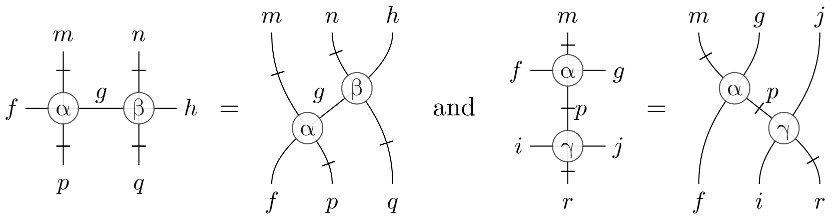


$$=\quad\begin{tikzcd}
& & \bullet \arrow[dl, dashed] \arrow[dr, dashed] \arrow[dd, dashed, "\alpha\odot\beta"] & & \\
& \bullet \arrow[ddl, "p_0"'] \arrow[ddr] \arrow[ddd, "\alpha"'] & & \bullet \arrow[ddl] \arrow[ddr, "q_1"] \arrow[ddd, "\beta"] & \\
& & \bullet \arrow[ddl, dashed] \arrow[ddr, dashed] & & \\
\bullet \arrow[dd, "f"'] & & \bullet \arrow[dd, "g"] & & \bullet \arrow[dd, "h"] \\
& \bullet \arrow[dl] \arrow[dr, "r_1"'] & & \bullet \arrow[dl, "s_0"] \arrow[dr] & \\
\bullet & & \bullet & & \bullet
\end{tikzcd}$$

Span double categories are in general not flat, yet their 2-dimensional structure is entirely determined by their 1-dimensional structure.

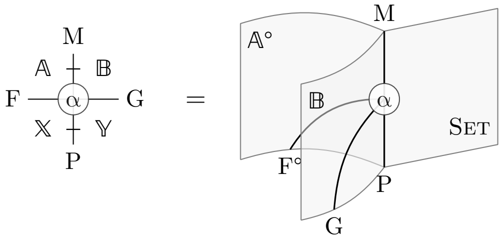
Example 2.3 (quintet double categories)
For a 2-category $\mathbb{C}$ we define the *quintet double category* $\mathrm{Q}\mathbb{C}$, where both edge categories are the category of objects and morphisms of $\mathbb{C}$. The set of squares ${}^{m}_{f}\Diamond^{g}_{n}$ in $\mathrm{Q}\mathbb{C}$ is the set of disks $m \cdot g \to f \cdot n$ in $\mathbb{C}$; so a square of $\mathrm{Q}\mathbb{C}$ is a disk of $\mathbb{C}$ together with a chosen binary factorization of its boundary morphisms. Composition in $\mathrm{Q}\mathbb{C}$ is the same as in $\mathbb{C}$, but keeping track of whether a morphism

is regarded as an arrow or a proarrow:



This makes $\mathbb{QC}$ a strict double category. Quintet double categories were introduced by Ehresmann in [Ehr63].

Example 2.4 (the double category of categories, functors, and profunctors) There is a double category $\textsc{Cat}$, where objects are (suitably small) categories, arrows are functors, and proarrows are *profunctors*; that is, a proarrow $\mathbb{A} \nrightarrow \mathbb{B}$ is a functor $\mathbb{A}^{\circ} \times \mathbb{B} \rightarrow \textsc{Set}$ (where this "$\textsc{Set}$" is the ordinary 1-category of sets and functions, our $\textsc{Set}_0$). We express the set determined by profunctor $\mathrm{M} : \mathbb{A} \nrightarrow \mathbb{B}$ acting on objects $\mathrm{A} : \mathbb{A}$ and $\mathrm{B} : \mathbb{B}$ using the notation "$\mathrm{M}\,(\mathrm{A} \dashrightarrow \mathrm{B})$", which we can think of as the set of M-*heteromorphisms* between A and B. Pairs of morphisms $a : \mathbb{A}\,(\mathrm{A}' \rightarrow \mathrm{A})$ and $b : \mathbb{B}\,(\mathrm{B} \rightarrow \mathrm{B}')$ determine a function $\mathrm{M}\,(a \dashrightarrow b) : \textsc{Set}\,(\mathrm{M}\,(\mathrm{A} \dashrightarrow \mathrm{B}) \rightarrow \mathrm{M}\,(\mathrm{A}' \dashrightarrow \mathrm{B}'))$. Squares of the double category $\textsc{Cat}$ are disks of the cartesian monoidal 2-category of categories, functors and natural transformations. In particular, a square $\alpha : {}^{\mathrm{M}}_{\mathrm{F}}\Diamond^{\mathrm{G}}_{\mathrm{P}}$ is a natural transformation $\alpha : (\mathbb{A}^{\circ} \times \mathbb{B} \rightarrow \textsc{Set})\,(\mathrm{M} \rightarrow (\mathrm{F}^{\circ} \times \mathrm{G}) \cdot \mathrm{P})$:



with component functions $\alpha(\mathrm{A}\,,\mathrm{B}) : \textsc{Set}\,(\mathrm{M}\,(\mathrm{A} \dashrightarrow \mathrm{B}) \rightarrow \mathrm{P}\,(\mathrm{FA} \dashrightarrow \mathrm{GB}))$.

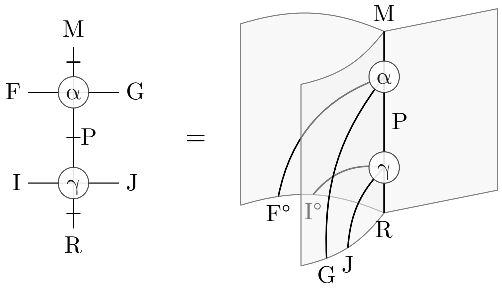
The composition of profunctors $\mathrm{M} : \mathbb{A} \nrightarrow \mathbb{B}$ and $\mathrm{N} : \mathbb{B} \nrightarrow \mathbb{C}$ is defined by a choice of coend in the formula $\mathrm{M} \odot \mathrm{N}\,(\mathrm{A} \dashrightarrow \mathrm{C}) := \int^{\mathrm{B}:\mathbb{B}} \mathrm{M}\,(\mathrm{A} \dashrightarrow \mathrm{B}) \times \mathrm{N}\,(\mathrm{B} \dashrightarrow \mathrm{C})$. Composition units are the homogeneous hom profunctors, with $\mathrm{U}\mathbb{A}\,(\mathrm{X} \dashrightarrow \mathrm{Y}) := \mathbb{A}\,(\mathrm{X} \rightarrow \mathrm{Y})$. For a thorough introduction to coends and their calculus see Loregian's [Lor19].

The coend quotients the sets of ordered pairs of heteromorphisms by the relation of simultaneous factorization by morphisms in the intermediate category: for $x : \mathrm{M}\,(\mathrm{A} \dashrightarrow \mathrm{B})$, $y : \mathrm{N}\,(\mathrm{B} \dashrightarrow \mathrm{C})$, $x' : \mathrm{M}\,(\mathrm{A} \dashrightarrow \mathrm{B}')$ and $y' : \mathrm{N}\,(\mathrm{B}' \dashrightarrow \mathrm{C})$, we have $(x\,,y) = (x'\,,y')$ in $\mathrm{M} \odot \mathrm{N}\,(\mathrm{A} \dashrightarrow \mathrm{C})$ just in case there is a $b : \mathbb{B}\,(\mathrm{B} \rightarrow \mathrm{B}')$ such that $\mathrm{M}\,(\mathrm{A} \dashrightarrow b)(x) = x'$ and $\mathrm{N}\,(b \dashrightarrow \mathrm{C})(y') = y$. Intuitively, we can think of this as an associative law $x \cdot (b \cdot y') = (x \cdot b) \cdot y'$, which we can depict as follows.

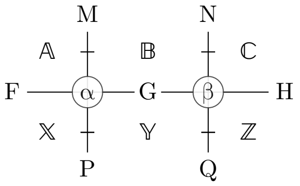
$$\begin{tikzcd}
 & \mathrm{B} \arrow[dd, "b"] \arrow[dr, "y"] & \\
\mathrm{A} \arrow[ur, "x"] \arrow[dr, "x'"'] & & \mathrm{C} \\
 & \mathrm{B}' \arrow[ur, "y'"'] &
\end{tikzcd} \tag{2.1}$$

Later, we will see that this can be interpreted literally as a commuting diagram in a certain category.

Square composition in the arrow dimension is by diagram pasting:



In the proarrow dimension, to compose squares $\alpha : {}^{M}_{F}\Diamond^{G}_{P}$ and $\beta : {}^{N}_{G}\Diamond^{H}_{Q}$, we first apply them independently as natural transformations $\alpha : M \to (F^{\circ} \times G) \cdot P$ and $\beta : N \to (G^{\circ} \times H) \cdot Q$, and then compose with the natural transformation $\int^{B:\mathbb{B}} P(F- \dashrightarrow GB) \times Q(GB \dashrightarrow H-) \to \int^{Y:\mathbb{Y}} P(F- \dashrightarrow Y) \times Q(Y \dashrightarrow H-)$ given by the unique morphism between the coends that factors components of the latter through those of the former as determined by the universal property of the coend:



$$
\begin{aligned}
= \qquad & M \odot N\,(A \dashrightarrow C) \\
:= \quad & \int^{B:\mathbb{B}} M\,(A \dashrightarrow B) \times N\,(B \dashrightarrow C) \\
\to \quad & \int^{B:\mathbb{B}} P\,(FA \dashrightarrow GB) \times Q\,(GB \dashrightarrow HC) \\
\to \quad & \int^{Y:\mathbb{Y}} P\,(FA \dashrightarrow Y) \times Q\,(Y \dashrightarrow HC) \\
=: \quad & P \odot Q\,(FA \dashrightarrow HC)
\end{aligned}
$$

3 Double Functors

When presented with a type of mathematical object we naturally seek to understand an appropriate notion of morphism between its instances. It is often the case that a natural notion of morphism between higher-dimensional categorical structures does not preserve the composition structure within them strictly, but rather up to canonical comparitor cells. The definition of lax double functor presented here is essentially that of [GP99] (where the unit naturality condition was omitted).

Definition 3.1 (lax double functor)
A *lax double functor* between double categories, $F : \mathbb{C} \to \mathbb{D}$, consists of a pair of (ordinary) functors $F_0 : \mathbb{C}_0 \to \mathbb{D}_0$ and $F_1 : \mathbb{C}_1 \to \mathbb{D}_1$ that are compatible with the structural boundary functors L and R in the sense that $F_1 \cdot L_{\mathbb{D}} = L_{\mathbb{C}} \cdot F_0$ and $F_1 \cdot R_{\mathbb{D}} = R_{\mathbb{C}} \cdot F_0$:

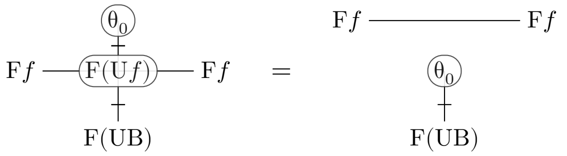
$$\begin{tikzcd}
\mathbb{C}_1 \arrow[r, "F_1"] \arrow[d, shift right, "L"'] \arrow[d, shift left, "R"] & \mathbb{D}_1 \arrow[d, shift right, "L"'] \arrow[d, shift left, "R"] \\
\mathbb{C}_0 \arrow[r, "F_0"'] & \mathbb{D}_0
\end{tikzcd}$$

and equipped with a system of coherent lax comparitors, θ. These can be characterized by natural transformations $\theta_0 : \mathrm{F}_0 \cdot \mathrm{U}_{\mathbb{D}} \to \mathrm{U}_{\mathbb{C}} \cdot \mathrm{F}_1$ and $\theta_2 : (\mathrm{F}_1 \times \mathrm{F}_1) \cdot \odot_{\mathbb{D}} \to \odot_{\mathbb{C}} \cdot \mathrm{F}_1$, whose components are proarrow disks

$$\begin{tikzcd}
\mathrm{FA} \arrow[r, "\mathrm{U}", "|" marking] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\theta_0(\mathrm{A})"] & \mathrm{FA} \arrow[d, "\mathrm{id}"] \\
\mathrm{FA} \arrow[r, "\mathrm{FU}"', "|" marking] & \mathrm{FA}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{FA} \arrow[r, "\mathrm{FM}", "|" marking] \arrow[d, "\mathrm{id}"'] & \mathrm{FB} \arrow[r, "\mathrm{FN}", "|" marking] \arrow[d, phantom, "\theta_2(\mathrm{M}\,,\mathrm{N})"] & \mathrm{FC} \arrow[d, "\mathrm{id}"] \\
\mathrm{FA} \arrow[rr, "\mathrm{F}(\mathrm{M}\odot\mathrm{N})"', "|" marking] & & \mathrm{FC}
\end{tikzcd}$$

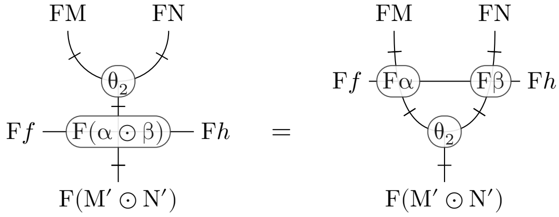
The naturality conditions amount to the following relations on the comparitor components.

unit naturality: for arrow $f : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$, the relation $\theta_0(\mathrm{A}) \cdot \mathrm{F}(\mathrm{U}f) = \mathrm{U}(\mathrm{F}f) \cdot \theta_0(\mathrm{B})$:



(3.1)

composition naturality: for consecutive $\mathbb{C}$-squares $\alpha : {}^{\mathrm{M}}_{f}\Diamond^{g}_{\mathrm{M}'}$ and $\beta : {}^{\mathrm{N}}_{g}\Diamond^{h}_{\mathrm{N}'}$, the relation $\theta_2(\mathrm{M}\,,\mathrm{N}) \cdot \mathrm{F}(\alpha \odot \beta) = (\mathrm{F}\alpha \odot \mathrm{F}\beta) \cdot \theta_2(\mathrm{M}'\,,\mathrm{N}')$:

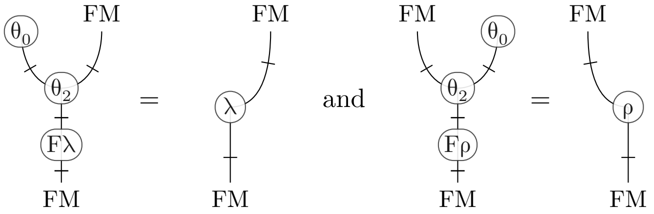


(3.2)

These lax comparitors are required to be compatible with the double category coherators in the following sense.

unitor compatibility: for $\mathbb{C}$-proarrow $\mathrm{M} : \mathrm{A} \nrightarrow \mathrm{B}$, the relations

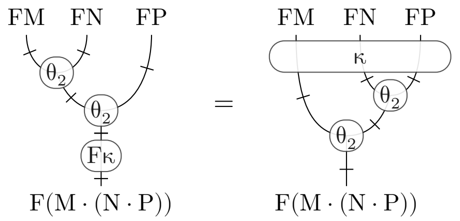
$$(\theta_0(\mathrm{A}) \odot \mathrm{FM}) \cdot \theta_2(\mathrm{A}\,,\mathrm{M}) \cdot \mathrm{F}(\lambda\mathrm{M}) = \lambda(\mathrm{FM}) \text{ and}$$
$$(\mathrm{FM} \odot \theta_0(\mathrm{B})) \cdot \theta_2(\mathrm{M}\,,\mathrm{B}) \cdot \mathrm{F}(\rho\mathrm{M}) = \rho(\mathrm{FM}) \ :$$



(3.3)

associator compatibility: for consecutive $\mathbb{C}$-proarrows $\mathrm{M} : \mathrm{A} \nrightarrow \mathrm{B}$, $\mathrm{N} : \mathrm{B} \nrightarrow \mathrm{C}$, and $\mathrm{P} : \mathrm{C} \nrightarrow \mathrm{D}$, the relation

$$(\theta_2(\mathrm{M}\,,\mathrm{N}) \odot \mathrm{FP}) \cdot \theta_2(\mathrm{M} \odot \mathrm{N}\,,\mathrm{P}) \cdot \mathrm{F}(\kappa(\mathrm{M}\,,\mathrm{N}\,,\mathrm{P})) \ =$$
$$\kappa(\mathrm{FM}\,,\mathrm{FN}\,,\mathrm{FP}) \cdot (\mathrm{FM} \odot \theta_2(\mathrm{N}\,,\mathrm{P})) \cdot \theta_2(\mathrm{M}\,,\mathrm{N} \odot \mathrm{P}) \ :$$

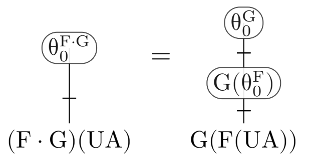


(3.4)

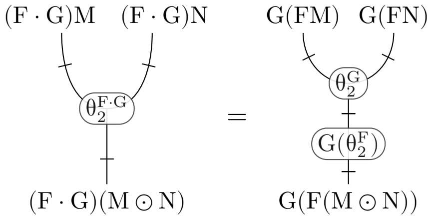
where we have momentarily made the associator explicit.

A double functor with comparitors that go the other way and are compatible with the coherator inverses is an *oplax double functor*. A lax or oplax double functor for which the nullary comparitor θ_0 is invertible is called *normal*. If the binary comparitor θ_2 is also invertible then we have a *pseudo double functor*. On the strict side, a double functor that strictly preserves identity proarrows is called *unitary*, and if it strictly preserves proarrow composites as well it is called *strict*.

Even lax double functors compose strictly, with composite comparitors given by



and



The strict identity double functors, $\mathrm{id}\mathbb{D} : \mathbb{D} \to \mathbb{D}$, which are the identity function on everything in sight, are composition units.

Next we consider some double functors that were presented by Grandis and Paré in [GP99].

Example 3.1
For any choice of pullbacks there is a double functor $R : \mathrm{SPAN}(\mathrm{SET}_0) \to \mathrm{SET}$ such that R_0 is the identity functor on the category SET_0. The functor R_1 sends a span $(p_0\,,p_1) : A \nrightarrow B$ to the relation that it determines in the sense that $\forall\, x \in \partial^- p_i \,.\, R(p_0\,,p_1)\,(p_0 x \dashrightarrow p_1 x)$; that is, the relation $R(p_0\,,p_1)$ relates $a \in A$ and $b \in B$ just in case there is an x in the span apex such that $p_0 x = a$ and $p_1 x = b$.

We will see soon (proposition 4.6) that R is a normal oplax double functor for purely formal reasons, but more is true. The relation determined by an identity span is the equality relation, so R is unitary. Any pullback $(p\,,q)$ of the inner legs of consecutive spans $(p_0\,,p_1) : A \nrightarrow B$ and $(q_0\,,q_1) : B \nrightarrow C$ is canonically isomorphic to the projection span $(\pi_0\,,\pi_1)$ from the set $W :=$

$\{(x\,,y) \in \mathrm{X} \times \mathrm{Y} \mid p_1 x = q_0 y\}$:

$$\begin{tikzcd}
 & & \mathrm{W} \arrow[dl, bend right, "\pi_0"'] \arrow[dr, bend left, "\pi_1"] & & \\
 & & \mathrm{Z} \arrow[u, dashed, "\cong"'] \arrow[dl, "p"'] \arrow[dr, "q"] & & \\
 & \mathrm{X} \arrow[dl, "p_0"'] \arrow[dr, "p_1"'] & & \mathrm{Y} \arrow[dl, "q_0"] \arrow[dr, "q_1"] & \\
\mathrm{A} & & \mathrm{B} & & \mathrm{C}
\end{tikzcd}$$

and we have

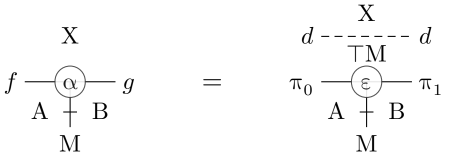
$$\begin{array}{rl}
 & (\mathrm{R}(p_0\,,p_1) \odot \mathrm{R}(q_0\,,q_1))\,(a \dashrightarrow c) \\
:= & \exists\, b \in \mathrm{B}\,.\ \mathrm{R}(p_0\,,p_1)\,(a \dashrightarrow b) \wedge \mathrm{R}(q_0\,,q_1)\,(b \dashrightarrow c) \\
:= & \exists\, b \in \mathrm{B}\,.\ (\exists\, x \in \mathrm{X}\,.\ p_0 x = a \wedge p_1 x = b) \wedge (\exists\, y \in \mathrm{Y}\,.\ q_0 y = b \wedge q_1 y = c) \\
\Rightarrow & \exists\,(x\,,y) \in \mathrm{W}\,.\ p_0 x = a \wedge q_1 y = c \\
\Rightarrow & \exists\, z \in \mathrm{Z}\,.\ p_0(pz) = a \wedge q_1(qz) = c \\
\Rightarrow & \mathrm{R}((p_0\,,p_1) \odot (q_0\,,q_1))\,(a \dashrightarrow c)
\end{array}$$

By relation extensionality, this together with the oplax comparitors makes R a strict double functor.

Example 3.2
The double functor R has a section $\mathrm{S} : \textsc{Set} \to \textsc{Span}(\textsc{Set}_0)$. The functor S_0 is again the identity on $\textsc{Set}_0$, while S_1 takes a relation $\mathrm{P} : \mathrm{A} \nrightarrow \mathrm{B}$ to its *tabulator*. This is the jointly monic projection span from the subset of the cartesian product containing the P-related pairs, $\{(a\,,b) \in \mathrm{A} \times \mathrm{B} \mid \mathrm{P}\,(a \dashrightarrow b)\}$. We will see soon (proposition 3.3) that S is a lax double functor for purely formal reasons. Because the tabulator of an equality relation is the diagonal set of ordered pairs and $\mathrm{A} \cong \{(a\,,a) \in \mathrm{A} \times \mathrm{A}\}$, it is moreover normal.

Grandis and Paré investigate the characterization of tabulators in arbitrary double categories in [GP99] and [GP17].

Definition 3.2 (proarrow tabulator)
In a double category $\mathbb{D}$, a *tabulator*[3] for a proarrow $\mathrm{M} : \mathrm{A} \nrightarrow \mathrm{B}$ is a universal square from the identity proarrow functor $\mathrm{U} : \mathbb{D}_0 \to \mathbb{D}_1$ to M. Explicitly, this constitutes an object $\top\mathrm{M}$, a span of arrows $\pi_0\mathrm{M} : \top\mathrm{M} \to \mathrm{A}$ and $\pi_1\mathrm{M} : \top\mathrm{M} \to \mathrm{B}$, and a square $\varepsilon\mathrm{M} : {}^{\mathrm{U}(\top\mathrm{M})}_{\pi_0\mathrm{M}}\Diamond^{\pi_1\mathrm{M}}_{\mathrm{M}}$ such that any square α from an identity proarrow to M factors uniquely through $\varepsilon\mathrm{M}$ by an identity arrow disk:



(3.5)

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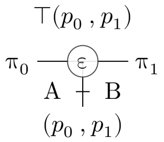
[3] In [GP99] Grandis and Paré call these "one-dimensional tabulators" and consider an additional two-dimensional universal property instrumental in defining double limits. In [GP17] they drop the qualifier "one-dimensional" and refer to these simply as "tabulators", as we shall do here.

A double category $\mathbb{D}$ has tabulators for all proarrows just in case the identity proarrow functor $\mathrm{U} : \mathbb{D}_0 \to \mathbb{D}_1$ has a right adjoint, in which case the square $\varepsilon\mathrm{M} : \mathbb{D}_1\,(\mathrm{U}(\top\mathrm{M}) \to \mathrm{M})$ is the adjunction counit component at M, and we have a natural bijection of hom sets given by

$$\frac{\alpha : \mathbb{D}_1\,(\mathrm{UX} \to \mathrm{M})}{d : \mathbb{D}_0\,(\mathrm{X} \to \top\mathrm{M})}.$$

All of the double categories considered in section 2 have tabulators for all proarrows, with Set as the motivating example. In a span double category, a tabulator for a span $(p_0\,, p_1) : \mathrm{A} \nrightarrow \mathrm{B}$ is its apex object P with



$$=$$

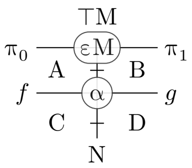
$$\begin{tikzcd} & \mathrm{P} \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] \arrow[dd, "\mathrm{id}"] & \\ \mathrm{P} \arrow[dd, "p_0"'] & & \mathrm{P} \arrow[dd, "p_1"] \\ & \mathrm{P} \arrow[dl, "p_0"] \arrow[dr, "p_1"'] & \\ \mathrm{A} & & \mathrm{B} \end{tikzcd}$$

In a quintet double category the domain object of a proarrow is a tabulator for it with $\varepsilon f : {}^{\mathrm{U}}_{\pi_0}\Diamond^{\pi_1}_{f} = \mathrm{id}f : \mathrm{id} \cdot f \to \mathrm{id} \cdot f$. In Cat a tabulator for a profunctor is a two-sided discrete fibration [Rie10] over its boundary known as its cocollage category [BS00]. The following result is from [GP17].

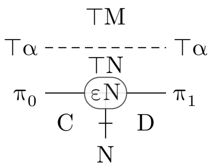
Proposition 3.3
If a double category $\mathbb{D}$ has pullbacks for arrows and tabulators for proarrows then there is a canonical lax double functor $\top : \mathbb{D} \to \mathrm{Span}(\mathbb{D}_0)$ that is the identity functor on $\mathbb{D}_0$ and takes proarrows to their tabulator spans.

Proof. (idea) The action of the double functor on squares and the structure of the lax comparitors all arise from the universal property of the tabulator.

- For arbitrary square $\alpha : {}^{\mathrm{M}}_{f}\Diamond^{g}_{\mathrm{N}}$ we have



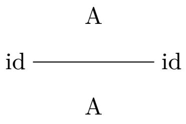
$$\overset{(3.5)}{=}$$



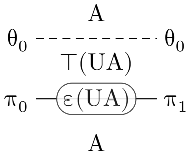
giving

$$\begin{tikzcd} & \top\mathrm{M} \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] \arrow[d, dashed, "\top\alpha"] & \\ \mathrm{A} \arrow[d, "f"'] & \top\mathrm{N} \arrow[dl, "\pi_0"] \arrow[dr, "\pi_1"'] & \mathrm{B} \arrow[d, "g"] \\ \mathrm{C} & & \mathrm{D} \end{tikzcd}$$

- Factoring an identity square by the tabulator we have



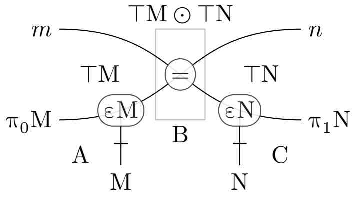
$$\overset{(3.5)}{=}$$



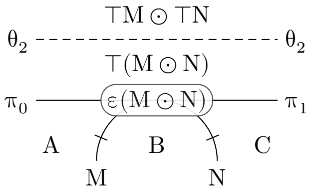
giving

$$\begin{tikzcd} & \mathrm{A} \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] \arrow[d, dashed, "\theta_0"] & \\ \mathrm{A} \arrow[d, "\mathrm{id}"'] & \top(\mathrm{UA}) \arrow[dl, "\pi_0"] \arrow[dr, "\pi_1"'] & \mathrm{A} \arrow[d, "\mathrm{id}"] \\ \mathrm{A} & & \mathrm{A} \end{tikzcd}$$

- Factoring a square to a composition of proarrows from the composition of their tabulator spans by the tabulator of the composite (where the highlighted arrow equality is the pullback of $\pi_1 M$ and $\pi_0 N$) we have



$\overset{(3.5)}{=}$



giving

$$\begin{tikzcd}
& & \top M \odot \top N \arrow[dl, "m"'] \arrow[dr, "n"] \arrow[dd, dashed, "\theta_2"] & & \\
& \top M \arrow[dl, "\pi_0"'] & & \top N \arrow[dr, "\pi_1"] & \\
A \arrow[d, "\mathrm{id}"'] & & \top(M \odot N) \arrow[dll, "\pi_0"] \arrow[drr, "\pi_1"'] & & C \arrow[d, "\mathrm{id}"] \\
A & & & & C
\end{tikzcd}$$

□

Example 3.3
We can extend the functor embedding of sets as discrete categories into a pseudo double functor $D : \textsc{Span}(\textsc{Set}_0) \to \textsc{Cat}$. A span $(p_0\,, p_1) : A \nrightarrow B$ gets sent to a profunctor with $D(p_0\,, p_1)\,(a \dashrightarrow b) = \{x \in \partial^- p_i \mid p_0 x = a \wedge p_1 x = b\}$; that is, the span's apex is regarded as the set of all heteromorphisms of a profunctor, and its legs as boundary assignment functions. A span morphism,

$$\begin{tikzcd}
& X \arrow[dl, "p_0"'] \arrow[dr, "p_1"] \arrow[dd, "\alpha"] & \\
A \arrow[dd, "f"'] & & B \arrow[dd, "g"] \\
& Y \arrow[dl, "q_0"] \arrow[dr, "q_1"'] & \\
C & & D
\end{tikzcd}$$

gets sent to itself, regarded as a (trivially natural) family of functions,

$$D\alpha(a\,, b) = \alpha\,|_{\{x \mid p_0 x = a \wedge p_1 x = b\}} : \textsc{Set}\,(\underbrace{D(p_0\,, p_1)\,(a \dashrightarrow b)}_{\{x \mid p_0 x = a \wedge p_1 x = b\}} \to \underbrace{D(q_0\,, q_1)\,(fa \dashrightarrow gb)}_{\{y \mid q_0 y = fa \wedge q_1 y = gb\}})$$

Example 3.4
The functor D_0 has as left adjoint the functor taking a category to its set of connected components. This extends to a normal oplax double functor $\pi_0 : \textsc{Cat} \to \textsc{Span}(\textsc{Set}_0)$. A profunctor $M : \mathbb{A} \nrightarrow \mathbb{B}$ goes to a span $(\partial^-\,, \partial^+) : \pi_0\mathbb{A} \nrightarrow \pi_0\mathbb{B}$ with apex set $\pi_0 M$, where a heteromorphism $x : M\,(A \dashrightarrow B)$ gets sent to its equivalence class $[x] \in \pi_0 M$ with $\partial^-[x] = [A] \in \pi_0\mathbb{A}$ and $\partial^+[x] = [B] \in \pi_0\mathbb{B}$, and such that for each $a : \mathbb{A}\,(A' \to A)$ and $b : \mathbb{B}\,(B \to B')$ we have $[x] = [M\,(a \dashrightarrow b)(x)]$. This is well defined because the existence of a and b imply the identifications $[A'] = [A]$ and $[B] = [B']$, in $\pi_0\mathbb{A}$ and $\pi_0\mathbb{B}$, respectively. A natural transformation $\alpha : M \to (F^\circ \times G) \cdot N$ goes to the function $\pi_0\alpha : \pi_0 M \to \pi_0 N$

such that for $x : \mathrm{M}(\mathrm{A} \dashrightarrow \mathrm{B})$ we have $\pi_0\alpha[x] = [\alpha(\mathrm{A}\,,\mathrm{B})(x)]$ and the following diagram commutes.

$$\begin{tikzcd}
 & \pi_0\mathrm{M} \arrow[dl, "\partial^-"'] \arrow[dr, "\partial^+"] \arrow[dd, "\pi_0\alpha"] & \\
\pi_0\mathbb{A} \arrow[dd, "\pi_0\mathrm{F}"'] & & \pi_0\mathbb{B} \arrow[dd, "\pi_0\mathrm{G}"] \\
 & \pi_0\mathrm{N} \arrow[dl, "\partial^-"] \arrow[dr, "\partial^+"'] & \\
\pi_0\mathbb{X} & & \pi_0\mathbb{Y}
\end{tikzcd}$$

The nullary comparitors $\theta_0\mathbb{C} : \pi_0(\mathbb{C}(- \to -)) \to \pi_0\mathbb{C}$ are the bijections of sets determined by the connected components. The oplax binary comparitors

$$\begin{tikzcd}
 & & \pi_0(\mathrm{M}\odot\mathrm{N}) \arrow[dll, "\partial^-"'] \arrow[drr, "\partial^+"] \arrow[d, dashed, "\theta_2"] & & \\
\pi_0\mathbb{A} \arrow[dd, "\mathrm{id}"'] & & \pi_0\mathrm{M}\odot\pi_0\mathrm{N} \arrow[dl] \arrow[dr] & & \pi_0\mathbb{C} \arrow[dd, "\mathrm{id}"] \\
 & \pi_0\mathrm{M} \arrow[dl] \arrow[dr] & & \pi_0\mathrm{N} \arrow[dl] \arrow[dr] & \\
\pi_0\mathbb{A} & & \pi_0\mathbb{B} & & \pi_0\mathbb{C}
\end{tikzcd}$$

act on consecutive heteromorphisms $x : \mathrm{M}(\mathrm{A} \dashrightarrow \mathrm{B})$ and $y : \mathrm{N}(\mathrm{B} \dashrightarrow \mathrm{C})$ by sending $[x \cdot y] \in \pi_0(\mathrm{M}\odot\mathrm{N})$ to its equivalence class in the pullback $\pi_0\mathrm{M}\odot\pi_0\mathrm{N} = \{(s\,,t) \in \pi_0\mathrm{M} \times \pi_0\mathrm{N} \mid \partial^+ s = \partial^- t\}$, where the consecutivity of x and y implies $\partial^+[x] = \partial^-[y]$ in $\pi_0\mathbb{B}$. We generally don't have comparitors going the other way. For example, if $\mathbb{A}$ and $\mathbb{C}$ are singleton categories, $\mathbb{B}$ is the walking arrow $i : 0 \to 1$, the profunctor $\mathrm{M} : \mathbb{A} \nrightarrow \mathbb{B}$ contains only one heteromorphism $x : \mathrm{M}(\mathrm{A} \dashrightarrow 1)$ and likewise $\mathrm{N} : \mathbb{B} \nrightarrow \mathbb{C}$ contains only $y : \mathrm{N}(0 \dashrightarrow \mathrm{C})$, then $\pi_0\mathrm{M} \odot \pi_0\mathrm{N}$ is a singleton set while $\pi_0(\mathrm{M}\odot\mathrm{N})$ is empty, so it is hopeless to seek a function from the former to the latter.

There is a construction to reify the heteromorphisms of a profunctor into arrows of a category [BS00].

Definition 3.4 (collage category)
The *collage category* of a profunctor $\mathrm{M} : \mathbb{A} \nrightarrow \mathbb{B}$ is a category $\mathrm{Col}\,\mathrm{M}$ with the following structure.

objects: the disjoint union of $\mathbb{A}$-objects and $\mathbb{B}$-objects,

arrows: the disjoint union of $\mathbb{A}$-homomorphisms, $\mathbb{B}$-homomorphisms, and M-heteromorphisms:

$$\mathrm{Col}\,\mathrm{M}(\mathrm{X} \to \mathrm{Y}) := \begin{cases} \mathbb{A}(\mathrm{X} \to \mathrm{Y}) & \text{if } \mathrm{X}, \mathrm{Y} : \mathbb{A} \\ \mathbb{B}(\mathrm{X} \to \mathrm{Y}) & \text{if } \mathrm{X}, \mathrm{Y} : \mathbb{B} \\ \mathrm{M}(\mathrm{X} \dashrightarrow \mathrm{Y}) & \text{if } \mathrm{X} : \mathbb{A} \text{ and } \mathrm{Y} : \mathbb{B} \\ \emptyset & \text{if } \mathrm{X} : \mathbb{B} \text{ and } \mathrm{Y} : \mathbb{A} \end{cases}$$

identities: inherited from $\mathbb{A}$ and $\mathbb{B}$,

composition: for consecutive ($\mathrm{Col}\,\mathrm{M}$)-arrows f and g,

- if both are $\mathbb{A}$-arrows then the composition $f \cdot g$ is inherited from $\mathbb{A}$,

- if both are $\mathbb{B}$-arrows then the composition $f \cdot g$ is inherited from $\mathbb{B}$,
- if $f : \mathbb{A}\,(\mathrm{A}' \to \mathrm{A})$ and $g : \mathrm{M}\,(\mathrm{A} \dashrightarrow \mathrm{B})$ then $f \cdot g : \mathrm{Col}\,\mathrm{M}\,(\mathrm{A}' \to \mathrm{B}) := \mathrm{M}\,(f \dashrightarrow \mathrm{B})(g) : \mathrm{M}\,(\mathrm{A}' \dashrightarrow \mathrm{B})$,
- if $f : \mathrm{M}\,(\mathrm{A} \dashrightarrow \mathrm{B})$ and $g : \mathbb{B}\,(\mathrm{B} \to \mathrm{B}')$ then $f \cdot g : \mathrm{Col}\,\mathrm{M}\,(\mathrm{A} \to \mathrm{B}') := \mathrm{M}\,(\mathrm{A} \dashrightarrow g)(f) : \mathrm{M}\,(\mathrm{A} \dashrightarrow \mathrm{B}')$.

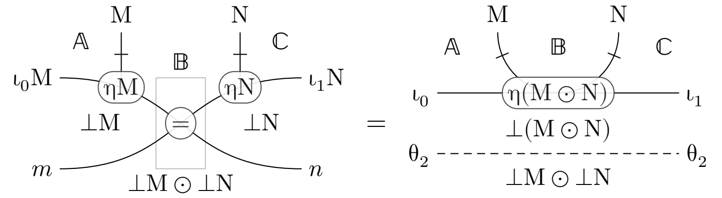
The associativity and unitality of composition in $\mathrm{Col}\,\mathrm{M}$ follow from the respective properties in $\mathbb{A}$ and $\mathbb{B}$ together with the functoriality of M. We refer to the inclusion functors as $\iota_0\mathrm{M} : \mathbb{A} \to \mathrm{Col}\,\mathrm{M}$ and $\iota_1\mathrm{M} : \mathbb{B} \to \mathrm{Col}\,\mathrm{M}$. The inclusion of profunctor heteromorphisms as collage arrows is a natural transformation between profunctors $\eta\mathrm{M} : (\mathbb{A}^\circ \times \mathbb{B} \to \textsc{Set})\,(\mathrm{M} \to \mathrm{Col}\,\mathrm{M}\,(\iota_0 - \to \iota_1 -))$, which is to say a square in ${}^{\mathrm{M}}_{\iota_0\mathrm{M}}\Diamond^{\iota_1\mathrm{M}}_{\mathrm{U}(\mathrm{Col}\,\mathrm{M})}$. Together, a collage category and its inclusions satisfy a universal property dual to that of (3.5), making the collage category a *cotabulator* for a profunctor.

Because we can form both pushouts of functors and collages of profunctors, by a result dual to proposition 3.3 there is a canonical oplax double functor $\perp : \textsc{Cat} \to \textsc{Cospan}(\textsc{Cat}_0)$ that is the identity functor on $\textsc{Cat}_0$ and takes profunctors to their collage cospans.

We can view the left half of equation (2.1) as a commuting diagram in the collage category $\perp\mathrm{M}$, and similarly its right half in $\perp\mathrm{N}$, but we cannot interpret the diagram as a whole in $\perp(\mathrm{M} \odot \mathrm{N})$. We can, however, interpret it in the category $\perp\mathrm{M} \odot \perp\mathrm{N}$, sometimes called the *gamut* of M and N, which is given by the following pushout in $\textsc{Cat}$ [Gra19]:

$$\begin{tikzcd}
\mathbb{A} \arrow[dr, hook, "\iota_0"'] & & \mathbb{B} \arrow[dl, hook', "\iota_1"'] \arrow[dr, hook, "\iota_0"] & & \mathbb{C} \arrow[dl, hook', "\iota_1"] \\
& \perp\mathrm{M} \arrow[dr, dashed, "m"'] & \wedge & \perp\mathrm{N} \arrow[dl, dashed, "n"] & \\
& & \perp\mathrm{M} \odot \perp\mathrm{N} & &
\end{tikzcd}$$

By the dual of (3.5), the universal property of the cotabulator determines a canonical functor from the collage to the gamut, $\theta_2(\mathrm{M}\ ,\ \mathrm{N}) : \perp(\mathrm{M} \odot \mathrm{N}) \to \perp\mathrm{M} \odot \perp\mathrm{N}$ as follows, where the highlighted arrow equality is the pushout above.



This functor is the comparitor component of the oplax cotabulator functor $\perp : \textsc{Cat} \to \textsc{Cospan}(\textsc{Cat}_0)$ and acts by inclusion.

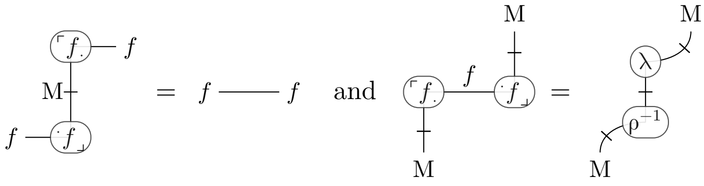
4 Connection Structures

Arrows and proarrows of a double category may be related to one another in various ways. Of particular interest is the case where a morphism of one sort has a “twin” of the other sort that acts like its reflection across the diagonal or antidiagonal of a square. Such a connection structure forms half of an adjunction structure in a precise way. Connection structures on pseudo double categories were investigated by Grandis and Paré in [GP04], their universal properties were described by Shulman in [Shu08], and their graphical calculus was explored by Myers in [Mye16].

Definition 4.1 (companion morphism)
In a double category, a parallel arrow $f : \mathrm{A} \to \mathrm{B}$ and proarrow $\mathrm{M} : \mathrm{A} \nrightarrow \mathrm{B}$ are *companions* if there are *connection squares*

$$\begin{tikzcd}
\mathrm{A} \arrow[r, "\mathrm{U}", "+"{marking}] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\ulcorner f_{\cdot}"] & \mathrm{A} \arrow[d, "f"] & & \mathrm{A} \arrow[r, "\mathrm{M}", "+"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "{}^{\cdot}f_{\lrcorner}"] & \mathrm{B} \arrow[d, "\mathrm{id}"] \\
\mathrm{A} \arrow[r, "\mathrm{M}"', "+"{marking}] & \mathrm{B} & \text{and} & \mathrm{B} \arrow[r, "\mathrm{U}"', "+"{marking}] & \mathrm{B}
\end{tikzcd}$$

satisfying the *companion laws* $\ulcorner f_{\cdot} \cdot {}^{\cdot}f_{\lrcorner} = \mathrm{U}(f)$ and $\ulcorner f_{\cdot} \odot {}^{\cdot}f_{\lrcorner} \cong \mathrm{id}(\mathrm{M})$:

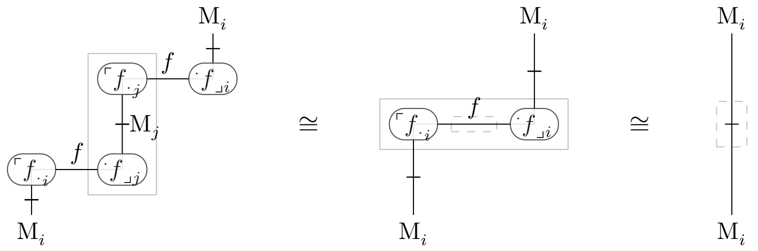


(4.1)

Companions, when they exist, are unique up to a canonical isomorphism.

Lemma 4.2 (uniqueness of companion morphisms)
If arrow $f : \mathrm{A} \to \mathrm{B}$ has companion proarrow M_0 with connection squares $\ulcorner f_{\cdot 0}$ and ${}^{\cdot}f_{\lrcorner 0}$ and also has companion proarrow M_1 with connection squares $\ulcorner f_{\cdot 1}$ and ${}^{\cdot}f_{\lrcorner 1}$ then (up to coherators) the proarrow disks $\ulcorner f_{\cdot 1} \odot {}^{\cdot}f_{\lrcorner 0} : \mathrm{M}_0 \to \mathrm{M}_1$ and $\ulcorner f_{\cdot 0} \odot {}^{\cdot}f_{\lrcorner 1} : \mathrm{M}_1 \to \mathrm{M}_0$ form an isomorphism.

Proof. For $\{i, j\} = \{0, 1\}$, we have:



□

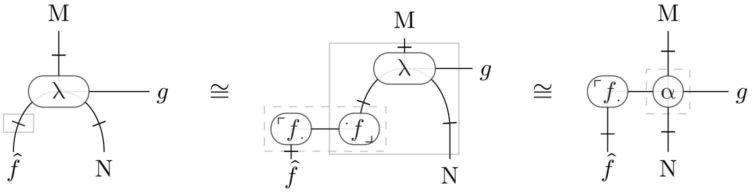
This justifies the convention of referring to *the* companion proarrow of an arrow f, which we will write as “$\hat{f}$”. Dually, we have the uniqueness of a companion arrow for a given proarrow. We will call an arrow or proarrow of a double category *companionable* if it has a companion morphism of the opposite sort.

This characterization of the companion relation, from [GP04], is *algebraic* in the sense of being presented by generators and relations. Equivalently, companionship can be characterized by universal properties, providing unique factorization by connection squares. That is the approach taken by Shulman in [Shu08].

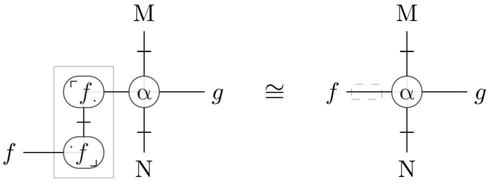
Lemma 4.3 (unique factorization by connection squares)
For any square $\alpha : {}^{\mathrm{M}}_{f}\Diamond^{g}_{\mathrm{N}}$,

- if f is companionable then there is a unique square $\lambda : {}^{\mathrm{M}}_{\mathrm{id}}\Diamond^{g}_{\hat{f}\odot\mathrm{N}}$ with $\lambda \cdot ({}^{\cdot}f_{\lrcorner} \odot \mathrm{N}) \cong \alpha$,
- if g is companionable then there is a unique square $\rho : {}^{\mathrm{M}\odot\hat{g}}_{f}\Diamond^{\mathrm{id}}_{\mathrm{N}}$ with $(\mathrm{M} \odot {}^{\ulcorner}g_{.}) \cdot \rho \cong \alpha$.

Proof. For the first of these dual results, observe that any square λ with the stated property must be (up to coherators) ${}^{\ulcorner}f_{.} \odot \alpha$:

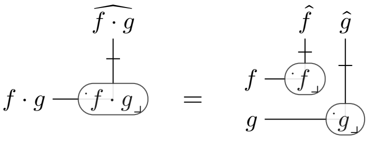


and that ${}^{\ulcorner}f_{.} \odot \alpha$ does indeed have the property:

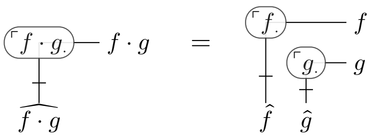


□

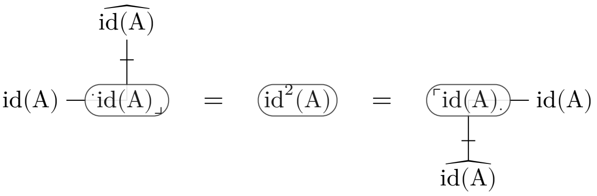
In a double category, the companionable morphisms of a given sort form a wide subcategory of the respective edge category with



,

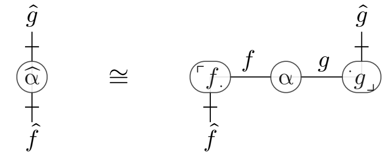


and



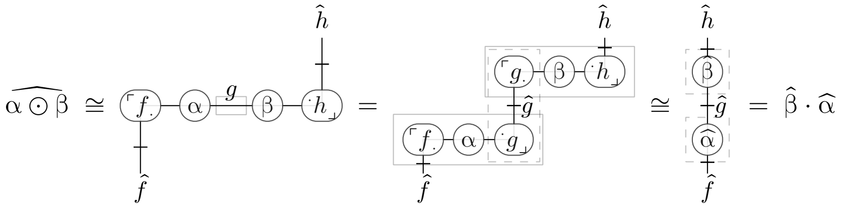
The companion relation links not just the 1-dimensional structure of a double category's edge categories, but the 2-dimensional structure of its disk bicategories as well.

Definition 4.4 (companion disk)
Given an arrow disk $\alpha : f \nrightarrow g$ with companionable boundary arrows, we define its *companion* proarrow disk $\widehat{\alpha} : \hat{g} \rightarrow \hat{f}$ to be (up to coherators) $\ulcorner f_{.} \odot \alpha \odot {}^{\cdot}g_{\lrcorner}$:



Dually, for a proarrow disk $\beta : \hat{g} \rightarrow \hat{f}$ with companionable boundary proarrows, its companion arrow disk is given by $\ulcorner g_{.} \cdot \beta \cdot {}^{\cdot}f_{\lrcorner} : f \nrightarrow g$.

Any disk with companionable boundary morphisms is itself companionable. Thus we obtain an equivalence between the sub-bicategories of the arrow- and proarrow disk bicategories with companionable morphisms, which is covariant on morphisms and contravariant on disks:

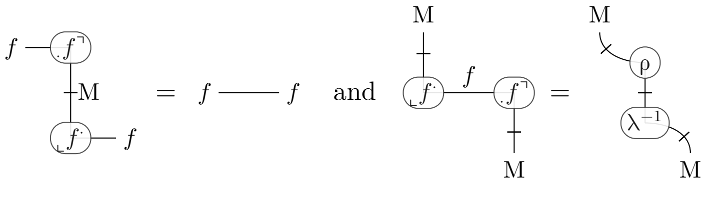


Intuitively, the companion relation provides a canonical way to reflect structure in a double category across the main diagonal of a square. We can reflect structure across a square's antidiagonal as well.

Definition 4.5 (conjoint morphism)
In a double category, an antiparallel arrow $f : \mathrm{A} \rightarrow \mathrm{B}$ and proarrow $\mathrm{M} : \mathrm{B} \nrightarrow \mathrm{A}$ are *conjoints* if there are *(co)connection squares*

$$\begin{tikzcd}
\mathrm{B} \arrow[r, "\mathrm{M}", "|" marking] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "{}_{\llcorner}f^{\cdot}"] & \mathrm{A} \arrow[d, "f"] \\
\mathrm{B} \arrow[r, "\mathrm{U}"', "|" marking] & \mathrm{B}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{A} \arrow[r, "\mathrm{U}", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, ".f^{\urcorner}"] & \mathrm{A} \arrow[d, "\mathrm{id}"] \\
\mathrm{B} \arrow[r, "\mathrm{M}"', "|" marking] & \mathrm{A}
\end{tikzcd}$$

satisfying the *conjoint laws* $.f^{\urcorner} \cdot {}_{\llcorner}f^{\cdot} = \mathrm{U}(f)$ and ${}_{\llcorner}f^{\cdot} \odot .f^{\urcorner} \cong \mathrm{id}(\mathrm{M})$:



(4.2)

All constructions and results involving companions have duals for conjoints, obtained by reversing the orientation of either $\mathbb{D}_0$ or $\mathbb{D}_1$ (but not both!). We will write the conjoint proarrow to a conjoinable arrow $f : \mathrm{A} \to \mathrm{B}$ as $\check{f} : \mathrm{B} \nrightarrow \mathrm{A}$, and the conjoint proarrow disk to a conjoinable arrow disk $\alpha : f \nrightarrow g$ as $\breve{\alpha} : \check{f} \to \check{g}$.

A double category in which every arrow has both a companion and a conjoint proarrow is a structure known variously as a *proarrow equipment* [Woo82], a *framed bicategory* [Shu08], or a *fibrant double category* [Ale18]. The results in this section are all standard in the literature of proarrow equipment. However, we wish to emphasize that they obtain *locally*: we need not assume the existence of companions or conjoints for all arrows in order to exploit their properties for those morphisms that do have them.

All of the double categories considered in section 2 have companions for all arrows, and all but the quintets have all conjoints as well.

Example 4.1 (connection structure in Set)
In the double category Set the companion to a function $f : \mathrm{A} \to \mathrm{B}$ is the relation given by $\forall\, a \in \mathrm{A}\ .\ \hat{f}\,(a \dashrightarrow fa) : \mathrm{A} \nrightarrow \mathrm{B}$, which is simply the result of regarding the correspondence given by f as a relation rather than as a function. The conjoint relation is given by $\forall\, a \in \mathrm{A}\ .\ \check{f}\,(fa \dashrightarrow a) : \mathrm{B} \nrightarrow \mathrm{A}$.

The relations $\hat{f}$ and $\check{f}$ form a *converse* pair that are logically equivalent for purely structural reasons. Recall that any relation $\mathrm{R} : \mathrm{A} \nrightarrow \mathrm{B}$ has a converse relation $\mathrm{R}^c : \mathrm{B} \nrightarrow \mathrm{A}$ such that $\mathrm{R}^c\,(b \dashrightarrow a) \iff \mathrm{R}\,(a \dashrightarrow b)$.

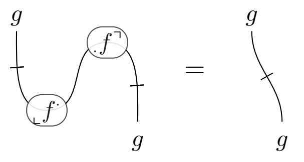
Example 4.2 (connection structure in span double categories)
In a span double category the companion to an arrow $f : \mathrm{A} \to \mathrm{B}$ is the span $(\mathrm{A}\,,f) : \mathrm{A} \nrightarrow \mathrm{B}$ with

$$\ulcorner f_{\cdot} \quad = \quad \begin{tikzcd} {} & \mathrm{A} \arrow[dl] \arrow[dr] \arrow[dd] & {} \\ \mathrm{A} \arrow[dd] & {} & \mathrm{A} \arrow[dd, "f"] \\ {} & \mathrm{A} \arrow[dl] \arrow[dr, "f"'] & {} \\ \mathrm{A} & {} & \mathrm{B} \end{tikzcd} \qquad \text{and} \qquad {}^{\cdot}f_{\lrcorner} \quad = \quad \begin{tikzcd} {} & \mathrm{A} \arrow[dl] \arrow[dr, "f"] \arrow[dd, "f"'] & {} \\ \mathrm{A} \arrow[dd, "f"'] & {} & \mathrm{B} \arrow[dd] \\ {} & \mathrm{B} \arrow[dl] \arrow[dr] & {} \\ \mathrm{B} & {} & \mathrm{B} \end{tikzcd}$$

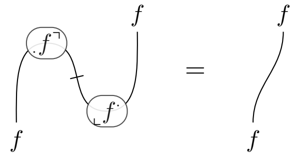
where the unlabeled endomorphisms are identities. The conjoint structure is just the horizontal reflection, with $\check{f} = (f\,,\mathrm{A})$.

Example 4.3 (connection structure in quintet double categories)
In a quintet double category the companion to an arrow $f : \mathrm{A} \to \mathrm{B}$ is f itself, but now regarded as a proarrow. The connection squares are both the identity disk on f, factored as the squares ${}^{\mathrm{U}}_{\mathrm{id}}\Diamond^{f}_{\hat{f}}$ and ${}^{\hat{f}}_{f}\Diamond^{\mathrm{id}}_{\mathrm{U}}$, respectively.

A conjoint for f would be a morphism $g : \mathrm{B} \to \mathrm{A}$, together with coconnection squares factoring ${}_{.}f^{\urcorner} : \mathrm{id}(\mathrm{A}) \to f \cdot g$ and ${}_{\llcorner}f^{\cdot} : g \cdot f \to \mathrm{id}(\mathrm{B})$, and satisfying



and



which is to say, a right adjoint to f.

Example 4.4 (connection structure in CAT)
In the double category CAT the companion and conjoint to a functor $\mathrm{F} : \mathbb{A} \to \mathbb{B}$ are the *represented* profunctors $\mathbb{B}\,(\mathrm{F}- \to -) : \mathbb{A}^{\circ} \times \mathbb{B} \to \textsc{Set}$ and $\mathbb{B}\,(- \to \mathrm{F}-) : \mathbb{B}^{\circ} \times \mathbb{A} \to \textsc{Set}$, respectively.

Note that the profunctors $\widehat{\mathrm{F}}$ and $\widecheck{\mathrm{F}}$ do not generally form a converse pair, as they do in SET. Any profunctor $\mathrm{M} : \mathbb{A} \nrightarrow \mathbb{B}$ does indeed have a converse M^{c}, but it lives in $\mathbb{B}^{\circ} \nrightarrow \mathbb{A}^{\circ}$:

$$\mathbb{A} \nrightarrow \mathbb{B} \;:=\; \mathbb{A}^{\circ} \times \mathbb{B} \to \textsc{Set} \;\cong\; \mathbb{B} \times \mathbb{A}^{\circ} \to \textsc{Set} \;=\; \mathbb{B}^{\circ\circ} \times \mathbb{A}^{\circ} \to \textsc{Set} \;=:\; \mathbb{B}^{\circ} \nrightarrow \mathbb{A}^{\circ}$$

In each of these examples, the companion-conjoint pair for a given arrow forms an adjunction in the proarrow disk bicategory. This is true in general, for any double category arrow f, it is the case that $\hat{f} \dashv \check{f}$, with the unit and counit formed by $\eta = \ulcorner f_{.} \odot {}_{.}f \urcorner$ and $\varepsilon = \llcorner f^{\cdot} \odot {}^{\cdot}f \lrcorner$, respectively.

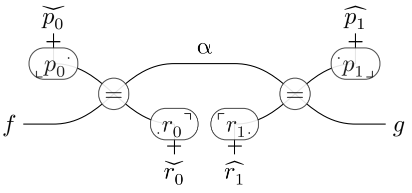
Just as tabulators give us a canonical double functor from a double category to its span double category (proposition 3.3), companions and conjoints give us a canonical double functor going the other way.

Proposition 4.6
If a double category $\mathbb{D}$ has companions, conjoints, and pullbacks for arrows then there is a canonical normal oplax double functor $\textsc{Span}(\mathbb{D}_0) \to \mathbb{D}$ that is the identity functor on $\mathbb{D}_0$ and takes a span $(p_0\,, p_1)$ to the composite proarrow $\widecheck{p_0} \odot \widehat{p_1}$.

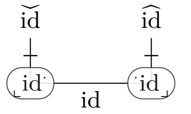
Proof. (idea)

- On squares the double functor acts as follows:

$$\begin{tikzcd}
 & \mathrm{P} \arrow[dl, "p_0"'] \arrow[dr, "p_1"] \arrow[dd, "\alpha"] & \\
\mathrm{A} \arrow[dd, "f"'] & & \mathrm{B} \arrow[dd, "g"] \\
 & \mathrm{R} \arrow[dl, "r_0"] \arrow[dr, "r_1"'] & \\
\mathrm{X} & & \mathrm{Y}
\end{tikzcd}
\qquad \longmapsto$$

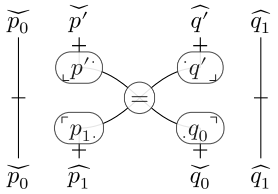


- For any object we obtain the following nullary oplax comparitor, which is invertible because $\llcorner\mathrm{id}^{\cdot}$ and ${}^{\cdot}\mathrm{id}\lrcorner$ are both just the identity square, id^2.



- For the composition of spans shown on the left, we obtain the binary oplax comparitor shown on the right, where the arrow equality is the commuting pullback square.

$$\begin{tikzcd}
 & & \mathrm{M} \arrow[dl, "p'"'] \arrow[dr, "q'"] & & \\
 & \mathrm{P} \arrow[dl, "p_0"'] \arrow[dr, "p_1"'] & & \mathrm{Q} \arrow[dl, "q_0"] \arrow[dr, "q_1"] & \\
\mathrm{A} & & \mathrm{B} & & \mathrm{C}
\end{tikzcd}$$



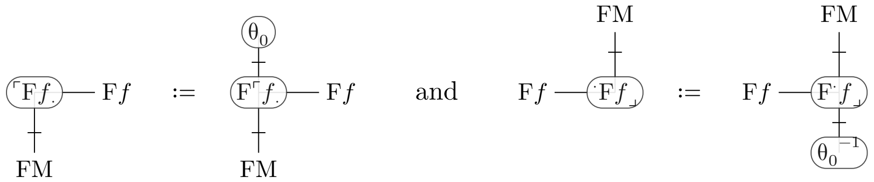
□

5 Connection Structure Preservation

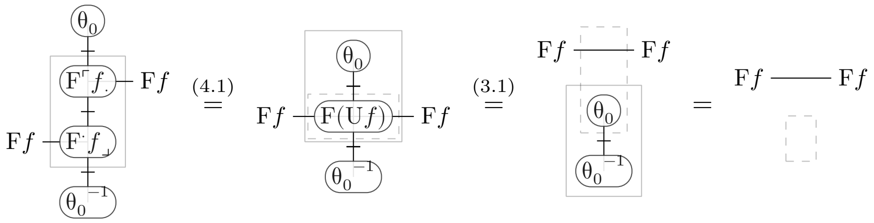
The connection structure afforded by companions and conjoints in a double category is quite rich, and it is natural to ask what sorts of double functors preserve it. It turns out that any lax or oplax double functor preserves this structure, so long as it weakly preserves identity proarrows; that is, so long as it is normal.

Proposition 5.1
Normal lax double functors preserve companions, in the sense that for any normal lax double functor $\mathrm{F} : \mathbb{C} \to \mathbb{D}$ if morphisms $f : \mathrm{A} \to \mathrm{B}$ and $\mathrm{M} : \mathrm{A} \nrightarrow \mathrm{B}$ are companions in $\mathbb{C}$ then the morphisms $\mathrm{F}f$ and FM are companions in $\mathbb{D}$.

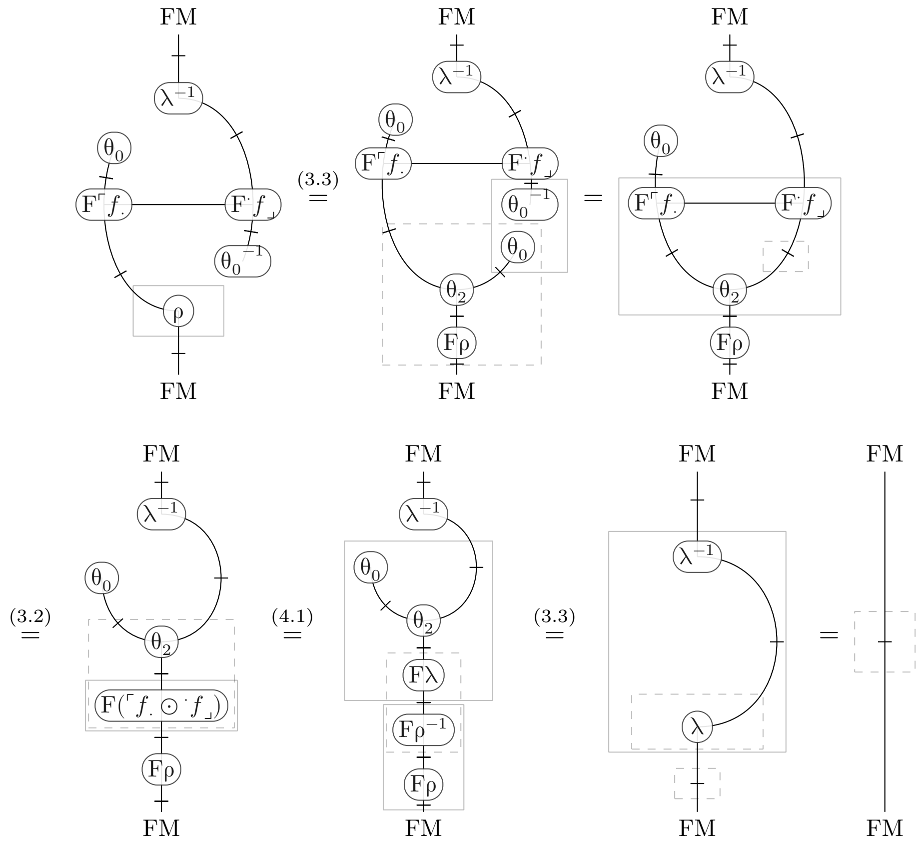
Proof. We take for connection squares of the F-images,



These connections satisfy the companion laws:



and



□

By horizontal reflection we obtain that normal lax double functors also preserve conjoints. By vertical reflection, normal oplax double functors preserve companions and conjoints as well.

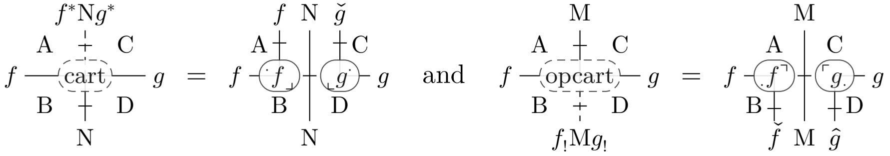
By the proposition we conclude that all of the double functors considered in section 3 preserve companions and conjoints.

6 Conclusion and Related Work

As mentioned in the introduction, the fact that normal lax or oplax double functors preserve the connection structure of double categories seems to be well known; thus the value (if any) of this write-up is not in the result itself, but rather in the direct, algebraic presentation and the perspicuity afforded by the graphical language of dual pasting diagrams. Here is a summary of the results on preservation of connection structure by double functors of which I am presently aware.

In [GP04] Grandis and Paré mention in passing that strictly unitary lax and oplax double functors preserve companions.

In [Shu08] Shulman develops the theory of framed bicategories from a non-algebraic perspective; that is, defined by universal properties rather than presented by generators and relations. There, he proves them equivalent to double categories with companions and conjoints for all arrows. In framed bicategories companions and conjoints are not primitive, but rather defined in terms of cartesian or opcartesian squares, which are universal squares specified by their boundary but missing one proarrow face. From our perspective, we can think of them as being given by



Shulman mentions that normal lax functors between framed bicategories preserve companions up to isomorphism, in the sense that there is an invertible proarrow disk $\mathrm{F}\hat{f} \to \widehat{\mathrm{F}f}$, where the companion of $\mathrm{F}f$ exists because all arrows in a framed bicategory are companionable. He also proves that any lax double functor between framed bicategories preserves cartesian squares. In [Ale18] Aleiferi generalizes this setting to arbitrary double categories and shows that any lax double functor with a lax left adjoint preserves cartesian squares.

Niefield shows in [Nie12] that for any double category $\mathbb{D}$ with arrow pullbacks, the identity functor on $\mathbb{D}_0$ extends to a normal oplax double functor $\mathrm{S{\scriptsize PAN}}(\mathbb{D}_0) \to \mathbb{D}$ just in case $\mathbb{D}$ has companions and conjoints for all arrows. From our perspective, since we know that $\mathrm{S{\scriptsize PAN}}(\mathbb{D}_0)$ itself has all companions and conjoints and that any normal oplax double functor must preserve them, the existence of such a double functor extending $\mathrm{id}(\mathbb{D}_0)$ implies that $\mathbb{D}$ must have them as well, giving us half of this result as a corollary. The other half is proposition 4.6.

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