Introduction to Categorical Semantics for Proof Theory (OPLSS 2015)

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Introduction

Category theory can be thought of as a sort of generalized set theory, where the primitive concepts are those of *set* and *function*, rather than *set* and *membership*. This shift of perspective allows categories to more directly describe many structures, even those that are not particularly set-like. The primitive concept of *set* generalizes to that of *object*, and *function* to *morphism*.

The only assumption that we make about these generalized functions is that they support a *composition structure*, whereby any configuration of compatible morphisms can be combined to yield a new morphism, and this operation is *associative* in the sense that the details of how we go about combining parts into a whole doesn't matter, only the configuration of those parts does.

This is reminiscent of many aspects of our physical world. When we build a castle out of Lego bricks, the order in which we assembled the bricks is not recorded anywhere in the finished product, only their configuration with respect to one another remains.

By beginning from very few assumptions, category theory permits a great deal of *axiomatic freedom*. Additional postulates (e.g. the axiom of choice) can then be selectively reintroduced in order to characterize a particular object theory of interest (e.g. set theory).

Because categorical characterizations are based on the concepts of object and morphism, they must describe their subjects *behaviorally* or *externally*, rather than *structurally* or *internally*: in category theory we can't pin down what the objects of our study actually *are*, only how they relate to one another. In this sense, category theory is the sociology of formal systems.

For example, we will see how we can characterize the cartesian product once and for all using a *universal property*. This allows us to describe cartesian products of sets, of groups, of topological spaces, of types, of propositions, and of countless other things, all in one fell swoop, rather than on a tedious case-by-case basis.

Proof theory itself is a formal system, and it has an associative composition structure: the order in which we build up the various sub-derivations of a proof does not affect the result. In this course, we will see how we can interpret the derivations of proof theory as the morphisms of a category, and how the meta-theory of that proof theory – which depends on the type of logic under consideration – determines the type of category in which these morphisms live.

It will emerge that there is a structural isomorphism, similar to the celebrated Curry-Howard correspondence, between proof theory and category theory (as well as between type theory and category theory, but that's a story for another time). Further, we will see that category theory provides the means to unify the proof theoretic behavior of the various logical connectives within a common algebraic framework.

Chapter 1

Basic Categories

1.1 Definition of a Category

Definition 1.1.0.1 (category) A **category** $\mathbb{C}$ consists of the following *data*:

- A collection of **objects**, $\mathbb{C}_0$.
  We write "$\mathrm{A} : \mathbb{C}$" to indicate that $\mathrm{A} \in \mathbb{C}_0$.
- A collection of **morphisms** or "arrows", $\mathbb{C}_1$.
  We write "$f :: \mathbb{C}$" to indicate that $f \in \mathbb{C}_1$.
- Two *boundary* functions, **domain**, "∂^-", and **codomain**, "∂^+", each mapping arrows to objects.
  For $\mathbb{C}$-objects A and B, we indicate the collection of $\mathbb{C}$-arrows with domain A and codomain B by "$\mathbb{C}\,(\mathrm{A} \to \mathrm{B})$", and call this collection "hom". When the category in question is obvious or irrelevant, we just write "$\mathrm{A} \longrightarrow \mathrm{B}$". We indicate that an arrow f is a member of this collection by writing "$f : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$" or "$f : \mathrm{A} \longrightarrow \mathrm{B}$".
- A function, **identity**, "id", mapping objects to arrows such that for any object A, $\mathrm{id}(\mathrm{A}) : \mathrm{A} \longrightarrow \mathrm{A}$.
- A partial function, **composition**, "$- \cdot -$", for pairs of arrows that is defined just in case the codomain of the first is equal to the domain of the second, in which case the composite arrow has the domain of the first and codomain of the second:

$$\text{if} \quad f : \mathrm{A} \longrightarrow \mathrm{B} \quad \text{and} \quad g : \mathrm{B} \longrightarrow \mathrm{C} \quad \text{then} \quad f \cdot g : \mathrm{A} \longrightarrow \mathrm{C}$$

This data is required to respect the following *relations*.

- **composition left unit law**: for an arrow $f : \mathrm{A} \longrightarrow \mathrm{B}$,

$$\mathrm{id}(\mathrm{A}) \cdot f \quad = \quad f$$

- **composition right unit law**: for an arrow $f : \mathrm{A} \longrightarrow \mathrm{B}$,

$$f \cdot \mathrm{id}(\mathrm{B}) \quad = \quad f$$

- **composition associative law**: for arrows $f : \mathrm{A} \longrightarrow \mathrm{B}$, $g : \mathrm{B} \longrightarrow \mathrm{C}$ and $h : \mathrm{C} \longrightarrow \mathrm{D}$,

$$(f \cdot g) \cdot h \quad = \quad f \cdot (g \cdot h)$$

By the associative law we may unambiguously write compositions without using brackets.

Definition 1.1.0.2 In order to avoid having to gratuitously name the boundaries of arrows, we will call a pair of arrows $f\,, g :: \mathbb{C}$:

- **coinitial** or a "span" if $\partial^-(f) = \partial^-(g)$,
- **coterminal** or a "cospan" if $\partial^+(f) = \partial^+(g)$,
- **composable** if $\partial^+(f) = \partial^-(g)$,
- **parallel** if both coinitial and coterminal, and
- **anti-parallel** if composable in both orders.

Additionally, we will call an arrow an **endomorphism** if it is composable with itself, and a list of arrows a **path** if they are serially composable, that is if $\partial^+(f_i) = \partial^-(f_{i+1})$ for the list $[f_0\,, \cdots, f_n]$.

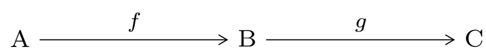
Remark 1.1.0.3 (applicative order composition) It is common to see the composition $f \cdot g$ written as "$g \circ f$". This can be useful when we want to apply a composite morphism to an argument in a category where a morphism is some sort of function. Then $(g \circ f)(x) = g(f(x))$, which coincides with our custom to write function application with the argument on the right. It may help to read "$f \cdot g$" as "f then g", and to read "$g \circ f$" as "g after f".

Remark 1.1.0.4 (dimensional promotion) It is often convenient to call the identity arrow on an object by the same name as the object, e.g. to write "A" in place of id(A). This will become useful later as we introduce more complex arrow constructions and concision becomes more of an issue.

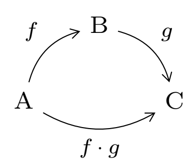
Remark 1.1.0.5 (unbiased presentation) There is an equivalent presentation of categories in terms of **unbiased composition**. There, instead of a single **binary composition** operation acting on a compatible *pair* of arrows, we have a length-indexed composition operation for *paths* of arrows (still with unit and associative laws). In this presentation, an identity morphism is a **nullary composition**, a morphism itself is a **unary composition**, and in general, any length n path of arrows has a unique composite. Although more cumbersome to axiomatize, an unbiased presentation of categories makes it easier to appreciate the idea at the heart of the definition: every composable configuration of things should have a unique composite.

1.2 Diagrams

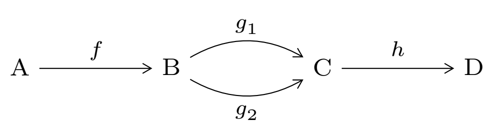
We think of an arrow as eminating *from* its domain and proceeding *to* its codomain. We may represent configurations of arrows in an intuitive graphical fashion using a **diagram**, such as:



We may represent equations between arrows using diagrams as well. We say that a diagram is **commuting** or "commutes" if the composites of *parallel* paths depicted in the diagram are equal. For example, the fact that all pairs of *composable* arrows have a unique composite gives us commuting composition triangles, such as:

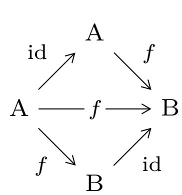


Commuting diagrams may be extended by pre- or post-composition of arrows, called **whiskering**, depicting the fact that equality of morphisms is a *congruence* with respect to composition: if $g_1 = g_2$ then $f \cdot g_1 = f \cdot g_2$ and $g_1 \cdot h = g_2 \cdot h$ whenever the composites are defined. The name comes from the fact that the arrows pre- or post-composed to the diagram look like whiskers:

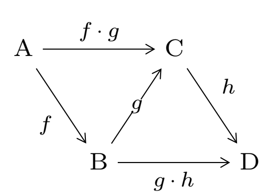


Pairs of commuting diagrams may also be combined along a common edge, called **pasting**, depicting the transitivity of equality: if $f_1 = f_2$ and $f_2 = f_3$ then $f_1 = f_3$.

We may express the *unit* and *associative* laws for composition with commuting diagrams as:



and



In the diagram for unitality (left), the triangles representing the left and right unit laws have been pasted along a common edge. In the diagram for associativity (right), each of the two composition triangles is whiskered by an arrow (h and f, respectively), and the resulting diagrams are pasted together along their common boundary ($f \cdot g \cdot h$).

In the graphical language of diagrams, any node representing an object may be duplicated and the two copies joined by an edge representing the appropriate identity morphism. Conversely, any edge representing an identity morphism may be collapsed, identifying the two nodes at its boundary, which necessarily represent the same object.

Except for the sake of emphasis, we generally omit composite arrows (including identitites, which are nullary composites) when drawing diagrams, because their existence may always be inferred. Notice that the associative law for composition is built into the graphical language of diagrams by the fact that there is no graphical representation for the bracketing of the arrows in a path.

In order to avoid gratuitously naming objects in diagrams, we will represent an anonymous object as a dot ("●"). Two such dots occurring in a diagram need not represent the same object.

1.3 Structured Sets as Categories

1.3.1 Discrete Categories

The most trivial possible category has nothing in it. It is called the **empty category**, and written "$\mathbb{0}$". Despite having completely uninteresting *structure*, we will see that this category nevertheless has a very interesting *property*.

Only slightly less trivially, we can consider a category with just a single object, call it "$\star$", and no arrows other than the required identity. This describes a **singleton category**, typically written "$\mathbb{1}$". This category will turn out to have a very interesting property as well.

Generalizing a bit, we can regard any *set* as a category. As a category, a set has its members as objects and no arrows other than the required identities. Such categories are called **discrete**.

1.3.2 Preorder Categories

A **preorder** is a reflexive and transitive binary relation on a set, typically written "$- \leq -$". We can interpret a preordered set $(\mathrm{P}, \leq)$ as a **preorder category** $\mathbb{P}$ in the following way:

objects $\mathbb{P}_0 \quad := \quad \mathrm{P}$

arrows $\mathbb{P}(x \to y) \quad := \quad \begin{cases} \{x \leq y\} & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases}$

identities $\mathrm{id}(x) \quad := \quad x \leq x$

composition $x \leq y \;\cdot\; y \leq z \quad := \quad x \leq z$

In other words, a preordered set is a category in which each hom collection is either empty, or else a singleton; and a hom is inhabited just in case its domain is less than or equal to its codomain according to the order relation.

A preorder need not have anything to do with our usual notion of order on a set. For example, the integers with the "divides" relation, $(\mathbb{Z}, |)$ is a perfectly good preordered set in which $2 \leq -2$, and also $-2 \leq 2$, and yet $2 \neq -2$.

In a preorder category the unit and associative laws of composition are trivially satisfied by the fact that all elements of a singleton or empty set are equal. In fact, every diagram in a preorder category must commute! Preorder categories are sometimes called "thin".

The simplest preorder category that is not discrete has two distinct objects and a single non-identity arrow from one to the other. It looks like this:

$$\bullet \longrightarrow \bullet$$

This category is called the **interval category**, and written "$\mathbb{I}$".

1.3.3 Monoid Categories

A **monoid** is a set M together with an associative binary operation "$- * -$" with neutral element "ε". We can interpret a monoid $(\mathrm{M}, *, \varepsilon)$ as a **monoid category** $\mathbb{M}$ in the following way:

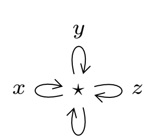
objects $\mathbb{M}_0 \quad := \quad \{\star\}$

arrows $\mathbb{M}(\star \to \star) \quad := \quad \mathrm{M}$

identities $\mathrm{id}(\star) \quad := \quad \varepsilon$

composition $x \cdot y \quad := \quad x * y$

Thus, a monoid becomes a category by "suspending" its elements into the hom collection of *endomorphisms* of an anonymous object, which I imagine looks something like this:



The unit and associative laws of composition are satisfied by the corresponding laws for the monoid operation.

If we wanted to make the simplest possible monoid category that is not discrete, we would have to think about what it means to be simple. We can begin by postulating a single non-identity arrow, $s : \star \longrightarrow \star$. But because s is an endomorphism, we must say what $s \cdot s$, $s \cdot s \cdot s$, and in general, $s^{(n)}$ are. One possibility is to introduce no relations. This gives us the free monoid on one generator, better known as $(\mathbb{N}\,, +\,, 0)$.

1.3.4 Categories of Propositions and Derivations

Although we are not yet in a position to give the details, we can sketch the outline of the sort of category we will use to interpret aspects of proof theory. The objects of such a category will be interpretations of propositions, and more generally, of propositional contexts. The arrows will be interpretations of derivations. For a derivation,

$$\frac{\frac{\Gamma}{\mathcal{D}}}{\mathrm{A}}$$

we will have $[\![\mathcal{D}]\!] : [\![\Gamma]\!] \longrightarrow [\![\mathrm{A}]\!]$.

Identity morphisms will be interpretations of *identity derivations*. These are derivations in which the inhabitant of a singleton set of premises is identically the conclusion; in other words, derivations comprised of no inference rules at all. Syntactically, they look just like propositions.

$$\mathrm{id}([\![\mathrm{A}]\!]) \;:=\; [\![\mathrm{A}]\!]$$

Compositions of morphisms will be interpretations of compositions of derivations, that is, of identifications of the conclusion of one derivation with a premise of another. We are not yet in a position to explain how this works in general, but schematically, the idea is this:

$$[\![\mathcal{D}]\!] \cdot [\![\mathcal{E}]\!] \;:=\; \text{interpretation of} \quad \frac{\frac{\frac{\Gamma}{\mathcal{D}}}{\mathcal{E}}}{\mathrm{A}}$$

We will fill in the details of how the interpretations of contexts and connectives work as we go along.

In addition to (structured) sets *as* categories, we also have categories *of* (structured) sets.

1.4 Categories of Structured Sets

1.4.1 The Category of Sets

There is a **category of sets**, called "Set", whose objects are sets and whose arrows are functions between them. Not surprisingly, we take function composition for the composition of arrows and identity functions for the identity arrows. That is, given composable functions f and g,

$$f \cdot g \quad := \quad \lambda x \,.\, g(f(x)) \qquad \text{and} \qquad \mathrm{id} \quad := \quad \lambda x \,.\, x$$

Composition of Set-morphisms is associative and unital precisely because composition of functions is (check this!).

1.4.2 The Category of Preorders

There is a **category of preorders**, called "PreOrd", that has *preordered sets* as objects and monotone (i.e. order-preserving) functions as arrows. Arrow composition is again function composition and the identity arrows are the identity functions.

In order to conclude that this is a category we (i.e. you) must check that the composition of monotone functions is again monotone, and that the identity functions are monotone. You just checked that function composition is associative and has identity functions as units, so since monotone functions are functions, you need not check associativity and unitality again for the special case.

1.4.3 The Category of Monoids

The **category of monoids**, Mon, has *monoids* as objects and monoid homomorphisms as arrows. A monoid homomorphism is a function between the underlying sets of the monoids that respects the operations and units:

$$\begin{array}{c} f : \textsc{Mon}\,((\mathrm{M}\,,*\,,\varepsilon) \to (\mathrm{N}\,,*'\,,\varepsilon')) \;\; := \;\; f : \textsc{Set}\,(\mathrm{M} \to \mathrm{N}) \\ \text{such that} \qquad f(x * y) = f(x) *' f(y) \qquad \text{and} \qquad f(\varepsilon) = \varepsilon' \end{array}$$

Abstract algebra provides a rich source of categories. Such categories generally have sets with some form of algebraic structure as objects and structure-preserving functions as arrows. In addition to that of monoids, we have the category of groups (Grp), of rings (Rng), of modules over a ring, and so on.

1.5 Categories of Categories

You may have noticed that we have interpreted some (hopefully) familiar mathematical structures (sets, preordered sets, monoids) *as* categories, but we have also described categories *of* these structures (Set, PreOrd, Mon). So these are in fact *categories of categories*! In each case, the objects comprise a sort of structured collection, and the arrows a mapping between these that respects the relevant structure.

Since categories themselves comprise a sort of structured collection, we may wonder whether we can identify a reasonable notion of arrow between categories, and thus define a category of categories. Indeed we can, so long as we heed a broad foundational restriction and avoid allowing a category of categories to be a member of itself. Otherwise, we leave ourselves open to paradoxes.

1.5.1 Functors

Recall that a category has collections of objects and arrows, together with an (associative and unital) composition structure. It is precisely this composition structure that we want an *arrow between categories* to preserve.

Definition 1.5.1.1 (functor) Given two *categories* $\mathbb{C}$ and $\mathbb{D}$, a **functor** F with domain $\mathbb{C}$ and codomain $\mathbb{D}$ consists of:

- a function on objects, $\mathrm{F}_0 : \mathbb{C}_0 \longrightarrow \mathbb{D}_0$,
- a function on arrows, $\mathrm{F}_1 : \mathbb{C}_1 \longrightarrow \mathbb{D}_1$,
  which respects the boundaries of arrows:

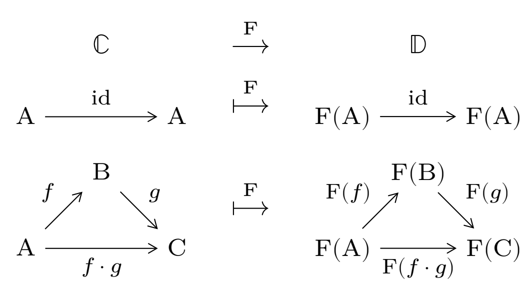
  $$f : \mathbb{C}\,(\mathrm{A} \to \mathrm{B}) \quad \longmapsto \quad \mathrm{F}_1(f) : \mathbb{D}\,(\mathrm{F}_0(\mathrm{A}) \to \mathrm{F}_0(\mathrm{B}))$$

  and which furthermore respects the composition structure:

  nullary composition $\quad \mathrm{F}_1(\mathrm{id}(\mathrm{A})) \quad = \quad \mathrm{id}(\mathrm{F}_0(\mathrm{A}))$

  binary composition $\quad \mathrm{F}_1(f \cdot g) \quad = \quad \mathrm{F}_1(f) \cdot \mathrm{F}_1(g)$

It is customary to drop the dimension subscripts on the constituent functions of a functor. We can represent the composition structure-preserving aspect of a functor diagrammatically as follows:



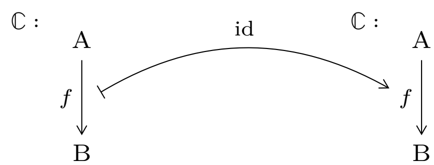
Equivalently, we could take the *unbiased* point of view and say that a functor respects the composition of arbitrary paths of arrows.

Functors provide a notion of *morphism of categories*. So we can ask about *their* composition structure as well. Because functors are defined in terms of functions, their composition structure is easy to define.

Definition 1.5.1.2 (identity functor) Given any category $\mathbb{C}$ we can define the **identity functor** on $\mathbb{C}$, $\mathrm{id}(\mathbb{C}) : \mathbb{C} \longrightarrow \mathbb{C}$, comprising identity functions on both objects and arrows:

$$(\mathrm{id}(\mathbb{C}))_0 \quad := \quad \mathrm{id}(\mathbb{C}_0) \qquad \text{and} \qquad (\mathrm{id}(\mathbb{C}))_1 \quad := \quad \mathrm{id}(\mathbb{C}_1)$$

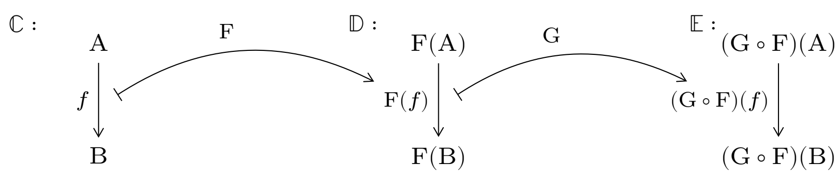
I imagine this as:



Definition 1.5.1.3 (functor composition) Given functors F from $\mathbb{C}$ to $\mathbb{D}$ and G from $\mathbb{D}$ to $\mathbb{E}$, we can define the **composition** $\mathrm{F} \cdot \mathrm{G}$ from $\mathbb{C}$ to $\mathbb{E}$, using the respective compositions on its object and arrow functions:

$$(\mathrm{F} \cdot \mathrm{G})_0 \quad := \quad \mathrm{F}_0 \cdot \mathrm{G}_0 \qquad \text{and} \qquad (\mathrm{F} \cdot \mathrm{G})_1 \quad := \quad \mathrm{F}_1 \cdot \mathrm{G}_1$$

I imagine this as:



Example 1.5.1.4 (forgetful functors) For a category of structured sets (such as a monoids, groups, ring or topological spaces) there is a **forgetful functor** to the category of sets, which disregards the structure and just retains the underlying set.

For instance, there is a forgetful functor $\mathrm{U} : \textsc{Mon} \longrightarrow \textsc{Set}$ that maps the monoid $(\mathbb{N}\,,+\,,0)$ to the set $\mathbb{N}$, and maps the monoid inclusion $(\mathbb{N}\,,+\,,0) \hookrightarrow (\mathbb{Z}\,,+\,,0)$ to the set inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$.

Lemma 1.5.1.5 (categories of categories) Given a collection of categories and functors between them, we can form the category having:

- the categories as objects

- paths in the functors as arrows
- identity functors as identity arrows
- functor composition as arrow composition

It is easy to check that the associative and unit laws of composition are satisfied.

Exercise 1.5.1.6 What is a functor:

- between *discrete categories*?
- between *preorder categories*?
- between *monoid categories*?

1.5.2 The Special Role of Sets

A collection is called "small" if it is a set. A category is called **small** if its collection of arrows – and hence, of objects – is small. There is a **category of small categories** and functors between them, called "Cat". Observe that it is *not* the case that Cat : Cat, because Cat_0 contains all the small *discrete categories*, i.e. the sets, and the collection of all sets is too large to be a set.

Often we don't care whether a category is (globally) small, but only that each of its hom collections is. A category is called **locally small** if for any pair of its objects, $A , B : \mathbb{C}$, the collection of arrows, $\mathbb{C}\,(A \to B)$ is small. Many categories of interest are locally small. In particular, Set and Cat are locally small (why?).

Unless otherwise specified, the categories that we encounter in this course will be locally small. Thus, we will stop being coy about what sort of "collection" a hom is, and refer instead to **hom sets**.

The fact that the collections of parallel arrows in a locally small category are sets puts the category Set in a privileged position. For example, if we fix an object $X : \mathbb{C}$, then we can define a function that, given any object $A : \mathbb{C}$, returns the set of arrows $\mathbb{C}\,(X \to A)$. This function extends to a functor:

Lemma 1.5.2.1 (representable functors) For each object of a locally small category, $X : \mathbb{C}$, there is a functor,

$$\begin{array}{rcl} & \underline{\mathbb{C}\,(X \to -)} & \\ \mathbb{C} & \longrightarrow & \text{Set} \\ A & \longmapsto & \mathbb{C}\,(X \to A) \\ f : A \longrightarrow B & \longmapsto & \mathbb{C}\,(X \to f) := - \cdot f : \mathbb{C}\,(X \to A) \longrightarrow \mathbb{C}\,(X \to B) \end{array}$$

known as a **representable functor**.

Unpacking this, it says that "$\mathbb{C}\,(X \to -)$" is the name of a functor from $\mathbb{C}$ to Set, that maps an object $A : \mathbb{C}$ to the *set* of arrows, $\mathbb{C}\,(X \to A)$, and maps an arrow $f : \mathbb{C}\,(A \to B)$ to the *function* that post-composes f to any arrow in $\mathbb{C}\,(X \to A)$, yielding an arrow in $\mathbb{C}\,(\mathrm{X} \to \mathrm{B})$. The notation "$- \cdot f$" is just syntactic sugar for $\lambda\,(a : \mathrm{X} \longrightarrow \mathrm{A})\;.\;a \cdot f$. The object $\mathrm{X} : \mathbb{C}$ is known as the "representing object" of this functor.

Proof. In order to show that $\mathbb{C}\,(\mathrm{X} \to -)$ is indeed a functor we must confirm that it preserves the composition structure:

nullary composition The idea is that post-composing an identity arrow does nothing, that is, it applies the identity function to the hom set. For $\mathrm{A} : \mathbb{C}$:

$$
\begin{array}{cl}
 & \mathbb{C}\,(\mathrm{X} \to \mathrm{id(A)}) \\
= & \text{[definition of representable functor]} \\
 & \lambda\, a \;.\; a \cdot \mathrm{id(A)} \\
= & \text{[composition unit law]} \\
 & \lambda\, a \;.\; a \\
= & \text{[definition of identity function]} \\
 & \mathrm{id}(\mathbb{C}\,(\mathrm{X} \to \mathrm{A}))
\end{array}
$$

binary composition Here, the idea is that post-composing a composite of arrows post-composes the first, and then post-composes the second to the result, that is, it composes the post-compositions. For $f : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$ and $g : \mathbb{C}\,(\mathrm{B} \to \mathrm{C})$:

$$
\begin{array}{cl}
 & \mathbb{C}\,(\mathrm{X} \to f \cdot g) \\
= & \text{[definition of representable functor]} \\
 & \lambda\, a \;.\; a \cdot f \cdot g \\
= & \text{[}\beta\text{-expansion]} \\
 & \lambda\, a \;.\; (\lambda\, b \;.\; b \cdot g)(a \cdot f) \\
= & \text{[}\beta\text{-expansion]} \\
 & \lambda\, a \;.\; (\lambda\, b \;.\; b \cdot g)((\lambda\, a \;.\; a \cdot f)(a)) \\
= & \text{[definition of function composition]} \\
 & (\lambda\, a \;.\; a \cdot f) \cdot (\lambda\, b \;.\; b \cdot g) \\
= & \text{[definition of representable functor]} \\
 & \mathbb{C}\,(\mathrm{X} \to f) \cdot \mathbb{C}\,(\mathrm{X} \to g)
\end{array}
$$

□

Because of the special role of the category of sets, the study of representable functors provides one of several, ultimately equivalent, ways of understanding categories. Due to our choice of emphasis and time constraints, it is not the one we will pursue here, but it is worth being aware of.

1.6 New Categories from Old

Now that we have met a few categories, let's look at some ways to create new categories out of them.

1.6.1 Product Categories

Given two categories, we may construct from them a new category whose constituent parts are just ordered pairs of the respective parts of the original categories.

Definition 1.6.1.1 (product category) For categories $\mathbb{C}_0$ and $\mathbb{C}_1$, define the **product category** $\mathbb{C}_0 \times \mathbb{C}_1$ to have the following structure:

objects $(\mathbb{C}_0 \times \mathbb{C}_1)_0 \; := \; \{(\mathrm{A}_0\,,\mathrm{A}_1) \mid \mathrm{A}_i : \mathbb{C}_i\}$

arrows $(\mathbb{C}_0 \times \mathbb{C}_1)\,((\mathrm{A}_0\,,\mathrm{A}_1) \to (\mathrm{B}_1\,,\mathrm{B}_1)) \; := \; \{(f_0\,,f_1) \mid f_i : \mathbb{C}_i\,(\mathrm{A}_i \to \mathrm{B}_i)\}$

identities $\mathrm{id}((\mathrm{A}_0\,,\mathrm{A}_1)) \; := \; (\mathrm{id}(\mathrm{A}_0)\,,\mathrm{id}(\mathrm{A}_1))$

composition $(f_0\,,f_1) \cdot (g_0\,,g_1) \; := \; (f_0 \cdot g_0\,,f_1 \cdot g_1)$

Note that we have not broken our promise to define products behaviorally, we have merely introduced a purely formal gadget called the "product of two categories" that we will make use of in subsequent constructions. Soon, however, we will be in a position to prove that this gadget does in fact have the universal property of a product.

1.6.2 Subcategories

Just as we may restrict our attention to a subset of a given set, we may single out a substructure of a category as well. However, since a category has more structure than a set, we must check that the substructure in question remains a category.

Definition 1.6.2.1 (subcategory) Given a category $\mathbb{C}$, we may take a **subcategory** $\mathbb{D}$ of $\mathbb{C}$, written "$\mathbb{D} \subseteq \mathbb{C}$" by taking for $\mathbb{D}_0$ a subcollection of $\mathbb{C}_0$ and for $\mathbb{D}_1$ a subcollection of $\mathbb{C}_1$, subject to the restrictions:

- if $f :: \mathbb{C}$ is in $\mathbb{D}_1$ then $\partial^-(f)$ and $\partial^+(f)$ are in $\mathbb{D}_0$.
- if $f\,,g :: \mathbb{C}$ are in $\mathbb{D}_1$ and are composable in $\mathbb{C}$ then $f \cdot g$ is in $\mathbb{D}_1$.
- if $\mathrm{A} : \mathbb{C}$ is in $\mathbb{D}_0$ then $\mathrm{id}(\mathrm{A})$ is in $\mathbb{D}_1$.

The composition structure of arrows when interpreted in $\mathbb{D}$ is the same as in $\mathbb{C}$.

The restrictions are necessary to ensure that the subcollections of $\mathbb{C}_0$ and $\mathbb{C}_1$ we choose do, in fact, form a category.

Whenever we have a subcategory $\mathbb{D} \subseteq \mathbb{C}$, we have also an **inclusion functor** $i : \mathbb{D} \longrightarrow \mathbb{C}$, written "$\mathbb{D} \hookrightarrow \mathbb{C}$", sending each object and arrow of $\mathbb{D}$ to itself, but now viewed as an object or arrow of $\mathbb{C}$.

1.6.3 Opposite Categories

Recall that each arrow in a category has two boundary objects, its *domain* and *codomain*. Systematically swapping these gives rise to an involutive relation on categories.

Definition 1.6.3.1 (opposite category) To any category $\mathbb{C}$, there corresponds an **opposite category**, $\mathbb{C}^\circ$ (pronounced "$\mathbb{C}$-op"), having:

objects ${\mathbb{C}^\circ}_0 \quad := \quad \mathbb{C}_0$

arrows $\mathbb{C}^\circ\,(\mathrm{A} \to \mathrm{B}) \quad := \quad \mathbb{C}\,(\mathrm{B} \to \mathrm{A})$

identities $\mathrm{id}(\mathrm{A}) :: \mathbb{C}^\circ \quad := \quad \mathrm{id}(\mathrm{A}) :: \mathbb{C}$

composition $f \cdot g :: \mathbb{C}^\circ \quad := \quad g \cdot f :: \mathbb{C}$

Exercise 1.6.3.2 Check that an opposite category satisfies the unit and associative laws of composition, and that the opposite of an opposite category is just the original category.

Despite being simple and purely formal, the op-construction is very useful. Because it is an *involution* (for any category $\mathbb{C}$, we have that $(\mathbb{C}^\circ)^\circ = \mathbb{C}$), op is called a **duality**.

For any categorical construction that we may perform, we can perform it in the opposite of a particular category. When viewed from the perspective of the original category, this gives rise to a **dual construction**. In many cases (including that of categorical logic) a construction and its dual may arise in the same category and interact in interesting ways.

Furthermore, any theorem that is true of a given category automatically has a **dual theorem** that is true of its opposite category. As a consequence, the dual of a theorem that is true of all categories is itself true of all categories!

Functors respect the op duality in the sense that whenever we have a functor $\mathrm{F} : \mathbb{C} \longrightarrow \mathbb{D}$, we automatically also have the functor $\mathrm{F}^\circ : \mathbb{C}^\circ \longrightarrow \mathbb{D}^\circ$. F° is really just the same functor as F, it merely lets the categories on its boundary imagine that their arrows are going the other way round.

A functor $\mathrm{F} : \mathbb{C}^\circ \longrightarrow \mathbb{D}$ is called a **contravariant functor** from $\mathbb{C}$ to $\mathbb{D}$. Among the most important contravariant functors one encounters are the contravariant representable functors:

Lemma 1.6.3.3 (contravariant representable functors) For any object of a locally small category, $X : \mathbb{C}$, there is a functor,

$$\begin{array}{rcl} & \underline{\mathbb{C}\,(- \to X)} & \\ \mathbb{C}^{\circ} & \longrightarrow & \textsc{Set} \\ A & \longmapsto & \mathbb{C}\,(A \to X) \\ f : A \longrightarrow B & \longmapsto & \mathbb{C}\,(f \to X) \coloneqq f \cdot - : \mathbb{C}\,(B \to X) \longrightarrow \mathbb{C}\,(A \to X) \end{array}$$

This is just an ordinary *representable functor* on the opposite category: $\mathbb{C}\,(- \to X) = \mathbb{C}^{\circ}\,(X \to -)$, because pre-composition in $\mathbb{C}$ is the same thing as post-composition in $\mathbb{C}^{\circ}$. For reasons that we won't dwell on, a contravariant representable functor is also known as a **representable presheaf**.

1.6.4 Arrow Categories

Definition 1.6.4.1 (arrow category) Given a category $\mathbb{C}$, we may derive from it another category, "$\mathbb{C}^{\to}$", known as the **arrow category** of $\mathbb{C}$ with the following structure:

objects ${\mathbb{C}^{\to}}_0 \quad \coloneqq \quad \mathbb{C}_1$

arrows $\mathbb{C}^{\to}\,(f \to g) \quad \coloneqq \quad \{(i\,,j) \mid i : \mathbb{C}\,(\partial^-(f) \to \partial^-(g))$ and $j : \mathbb{C}\,(\partial^+(f) \to \partial^+(g))$ such that $i \cdot g = f \cdot j\}$

identities $\mathrm{id}(f) \quad \coloneqq \quad (\mathrm{id}(\partial^-(f))\,, \mathrm{id}(\partial^+(f)))$

composition $(i\,,j) \cdot (k\,,l) \quad \coloneqq \quad (i \cdot k\,, j \cdot l)$

In a bit more detail, the objects of $\mathbb{C}^{\to}$ are the arrows of $\mathbb{C}$. Given $\mathbb{C}^{\to}$-objects, $f : \mathbb{C}\,(A \to B)$ and $g : \mathbb{C}\,(C \to D)$, a $\mathbb{C}^{\to}$-arrow from f to g is a pair of $\mathbb{C}$-arrows, $i : \mathbb{C}\,(A \to C)$ and $j : \mathbb{C}\,(B \to D)$ that form a commuting square with f and g in $\mathbb{C}$:

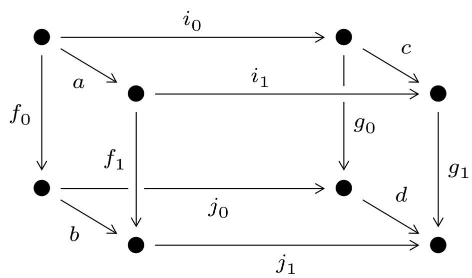
$$\begin{array}{ccc} A & \xrightarrow{i} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[j]{} & D \end{array} \tag{1.1}$$

Identity $\mathbb{C}^{\to}$-arrows are the commuting $\mathbb{C}$-squares with two opposite sides the same arrow and the other two opposite sides identity arrows. Composition in $\mathbb{C}^{\to}$ is the *pasting* of commuting squares in $\mathbb{C}$:

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[\mathrm{id}]{} & B \end{array} \qquad\qquad \begin{array}{ccccc} A & \xrightarrow{i} & C & \xrightarrow{k} & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ B & \xrightarrow[j]{} & D & \xrightarrow[l]{} & F \end{array}$$

The unit and associativity laws of composition are satisfied in $\mathbb{C}^{\to}$ as a consequence of their holding in $\mathbb{C}$ (you should check this). So the arrows of $\mathbb{C}^{\to}$ are the commuting squares of $\mathbb{C}$ (with each commuting $\mathbb{C}$-square represented twice). This tells us something about the 2-dimensional structure of $\mathbb{C}$, namely, which of its squares commute.

We can iterate this construction to explore yet higher-dimensional structure of $\mathbb{C}$. One dimension up, the category $(\mathbb{C}^{\rightarrow})^{\rightarrow}$ has as objects $\mathbb{C}^{\rightarrow}$-arrows (i.e. $\mathbb{C}$-commuting squares) and as arrows $\mathbb{C}^{\rightarrow}$-commuting squares. Let's see what these ought to be. A nice way to think about it is to take diagram (1.1) and imagine that it's actually a 3-dimensional cube that we happen to be seeing orthographically along one face. If we shift our perspective slightly, we will see the following:



(1.2)

We begin with $(\mathbb{C}^{\rightarrow})^{\rightarrow}$-objects f and g, which are actually the $\mathbb{C}^{\rightarrow}$-arrows from a to b and from c to d, respectively. These, in turn, are the $\mathbb{C}$-commuting squares shown on the left and right of diagram (1.2). Now $(\mathbb{C}^{\rightarrow})^{\rightarrow}$-arrows between these will be $\mathbb{C}^{\rightarrow}$-arrows between their domains and codomains, i and j, which are the $\mathbb{C}$-commuting squares shown on the top and bottom. But there is also the condition that $i \cdot g = f \cdot j$ in $\mathbb{C}^{\rightarrow}$. Composition in $\mathbb{C}^{\rightarrow}$ is pasting in $\mathbb{C}$, and equality of arrows in $\mathbb{C}^{\rightarrow}$ is just pairwise equality in $\mathbb{C}$. So we need that $i_0 \cdot g_0 = f_0 \cdot j_0$ and $i_1 \cdot g_1 = f_1 \cdot j_1$ in $\mathbb{C}$, making the back and front faces commute. In other words, the top and bottom commuting $\mathbb{C}$-squares form a $(\mathbb{C}^{\rightarrow})^{\rightarrow}$-arrow between the left and right commuting $\mathbb{C}$-squares just in case the front and back $\mathbb{C}$-squares commute as well. Then all the paths shown in diagram (1.2) commute. So $(\mathbb{C}^{\rightarrow})^{\rightarrow}$-arrows are $\mathbb{C}$-commuting cubes.

The arrow category construction provides us with three important functors, that in a sense "mediate between dimensions". These are the **domain**, **codomain** and **reflexivity** functors:

$$
\begin{array}{rcl}
 & \underline{dom} & \\
\mathbb{C}^{\rightarrow} & \longrightarrow & \mathbb{C} \\
f :: \mathbb{C} & \longmapsto & \partial^{-}(f) \\
(i\,,j) & \longmapsto & i
\end{array}
\qquad
\begin{array}{rcl}
 & \underline{cod} & \\
\mathbb{C}^{\rightarrow} & \longrightarrow & \mathbb{C} \\
f :: \mathbb{C} & \longmapsto & \partial^{+}(f) \\
(i\,,j) & \longmapsto & j
\end{array}
$$

$$
\begin{array}{rcl}
 & \underline{refl} & \\
\mathbb{C} & \longrightarrow & \mathbb{C}^{\rightarrow} \\
\mathrm{A} & \longmapsto & \mathrm{id(A)} \\
f & \longmapsto & (f\,,f)
\end{array}
$$

We will see later that these functors play an important role in the higher-dimensional structure of categories, but for now we will use the codomain functor to construct another important new category from old.

1.6.5 Slice Categories

Definition 1.6.5.1 (slice category) Given a category $\mathbb{C}$ and object $A : \mathbb{C}$, there is a category, "$\mathbb{C}/A$" called the **slice category** of $\mathbb{C}$ over A, with the following structure:

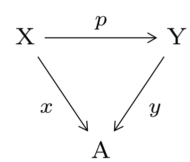
objects $(\mathbb{C}/A)_0 \quad := \quad \{f :: \mathbb{C} \mid \partial^+(f) = A\}$

arrows $\mathbb{C}/A\,(f \to g) \quad := \quad \{i : \mathbb{C}\,(\partial^-(f) \to \partial^-(g)) \mid i \cdot g = f\}$

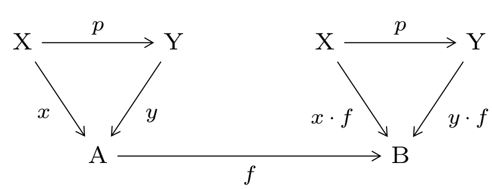
Identities and composition are inherited from $\mathbb{C}$.

The slice category $\mathbb{C}/A$ is a subcategory of the arrow category $\mathbb{C}^{\to}$. It contains just those objects and arrows that the *cod* functor sends to A and to id(A), respectively.

I imagine the arrows of $\mathbb{C}/A$ as the bases of inverted triangles with their vertices anchored at A:



Composing the sides of such triangles with an arrow $f : A \longrightarrow B$ lets us move the anchor to B:



Lemma 1.6.5.2 (post-composition functor) Every arrow $f : \mathbb{C}\,(A \to B)$ determines a functor:

$$\begin{array}{rcl} & \underline{f_!} & \\ \mathbb{C}/A & \longrightarrow & \mathbb{C}/B \\ x : \mathbb{C}\,(X \to A) & \longmapsto & x \cdot f \\ p : \mathbb{C}\,(X \to Y) & \longmapsto & p \end{array}$$

Chapter 2

Behavioral Reasoning

A fundamental question we must address when studying any kind of formal system is when two objects with distinct presentations should be considered to be essentially the same. We can ask this question about sets, groups, topological spaces, λ-terms, and even categories.

Certainly, whatever relation we choose should be an equivalence relation and should be a congruence for certain operations, but beyond that, general guidelines are hard to come by.

For example, we consider two sets to be essentially the same if there is a **bijection** between them, that is, if there is an injective and surjective function from one to the other. Recall that a function $p : \mathrm{X} \longrightarrow \mathrm{Y}$ is an **injection** if it "doesn't collapse any elements of its domain":

$$\forall\ x\ , y \in \mathrm{X}\ .\ p(x) = p(y)\ \supset\ x = y$$

and is a **surjection** if it "doesn't miss any elements of its codomain":

$$\forall\ y \in \mathrm{Y}\ .\ \exists\ x \in \mathrm{X}\ .\ p(x) = y$$

We can't translate such element-wise definitions directly to the language of categories because the objects of a category need not be structured sets. So we must find equivalent *behavioral* characterizations.

2.1 Monic and Epic Morphisms

2.1.1 Monomorphisms

In the case of injections, we can do this by rephrasing the property so that instead of talking about the *image* under p of two points of X, we talk about the *composition* with p of two parallel functions into X. So a function p is monic if:

$$\forall\ f\ , g : \mathrm{W} \to \mathrm{X}\ .\ \forall\ w \in \mathrm{W}\ .\ (p \circ f)(w) = (p \circ g)(w)\ \supset\ f(w) = g(w)$$

It may seem that we've just made things worse by introducing two extraneous functions, but now we can use the fact that two functions are equal just in case they agree on all points to rephrase this again, doing away with the points entirely. So a function p is monic if:

$$\forall\ f\ , g : \mathrm{W} \to \mathrm{X}\ .\ f \cdot p = g \cdot p\ \supset\ f = g$$

This is a behavioral characterization of injective function that can be stated for any category.

Definition 2.1.1.1 (monomorphism) An arrow $m :: \mathbb{C}$ is a **monomorphism** (or "monic") if it is *post-cancelable*. That is, if for any arrows $f\ , g :: \mathbb{C}$,

$$f \cdot m \quad = \quad g \cdot m \qquad \text{implies} \qquad f \quad = \quad g$$

Notice that we are being a bit economical here: in order for f and g to be composable with m, they must be *coterminal*, and in order for their composites with m to be equal, they must also be *coinitial*. So the definition is only applical to *parallel* f and g composable with m, but all that can be inferred.

In diagrams, monomorphisms are conventionally drawn with a tailed arrow: "$\rightarrowtail$".

Lemma 2.1.1.2 (monics and composition)

- Identity morphisms are monic.
- Composites of monics are monic.
- If the composite $m \cdot n$ is monic then so is m.

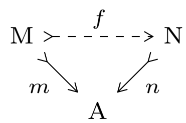
Proof.

$$\begin{array}{ll} & f \cdot \mathrm{id} \ = \ g \cdot \mathrm{id} \\ \Rightarrow & [\text{unit law}] \\ & f \ = \ g \end{array} \qquad \begin{array}{ll} & f \cdot m \cdot n \ = \ g \cdot m \cdot n \\ \Rightarrow & [n \text{ is monic}] \\ & f \cdot m \ = \ g \cdot m \\ \Rightarrow & [m \text{ is monic}] \\ & f \ = \ g \end{array} \qquad \begin{array}{ll} & f \cdot m \ = \ g \cdot m \\ \Rightarrow & [\text{whiskering}] \\ & f \cdot m \cdot n \ = \ g \cdot m \cdot n \\ \Rightarrow & [m \cdot n \text{ is monic}] \\ & f \ = \ g \end{array}$$

□

Remark 2.1.1.3 (subobjects) If we take the *subcategory* of a *slice category* containing just the monomorphisms then we get a *preorder*: given monics $m\,,n$:

$\mathbb{C}/\mathrm{A}$, we say that $m \leq n$ just in case there is an $f : \mathbb{C}/\mathrm{A}\,(m \rightarrow n)$.



Such an f, if it exists, is unique because n is monic, and is itself monic by the preceding lemma.

The monics into an object behave very much like the partial order of subsets of a set, in fact, they are known as (representatives of) **subobjects**. This is the beginning of a branch of categorical logic known as **topos theory**, where subobjects are used to interpret predicates. However, it is not the approach that we will pursue here because by using a preorder to interpret the entailment relation on propositions we would sacrifice *proof relevance*: we could say *that* one proposition entails another, but not *why* it does so.

2.1.2 Epimorphisms

Using the *op duality*, we can define the property dual to that of being monic. You should check that this amounts to the following:

Definition 2.1.2.1 (epimorphism) An arrow $e :: \mathbb{C}$ is an **epimorphism** (or "epic") if it is *pre-cancelable*. That is, if for any arrows $f\,,g :: \mathbb{C}$,

$$e \cdot f \quad = \quad e \cdot g \qquad \text{implies} \qquad f \quad = \quad g$$

This corresponds to the fact that a surjective function doesn't miss any points in its codomain, so if $p : \mathrm{X} \longrightarrow \mathrm{Y}$ is surjective then for any parallel $f\,,g : \mathrm{Y} \longrightarrow \mathrm{Z}$,

$$(\forall x \in \mathrm{X}\;.\;(f \circ p)(x) = (g \circ p)(x)) \quad \supset \quad (\forall y \in \mathrm{Y}\;.\;f(y) = g(y))$$

Eliminating the points gives us the definition of epimorphism.

In diagrams, epimorphisms are conventionally drawn with a double-headed arrow: "$\twoheadrightarrow$".

Exercise 2.1.2.2 State and prove the *dual theorems* to those in lemma 2.1.1.2.

In the category Set a function is injective just in case it is monic, and surjective just in case it is epic. In many categories of "structured sets" (e.g. Mon) the monomorphisms are exactly the injective homomorphisms. For instance, the inclusion $(\mathbb{N}\,,+\,,0) \hookrightarrow (\mathbb{Z}\,,+\,,0)$ in the category Mon is a monomorphism. It turns out to be an epimorphism as well, despite not being surjective on its underlying set. So, unlike the situation in Set, in an arbitrary category the existence of a monic and epic morphism between two objects does not suffice to ensure that they are *essentially the same*.

Monic and epic morphisms have other unsatisfactory properties, for example, they are not necessarily preserved by functors (the existence of a *forgetful functor* from monoids to sets, together with the last result implies this).

2.2 Split Monic and Epic Morphisms

Definition 2.2.0.3 (split monomorphism) An arrow s is a **split monomorphism** (or "split monic") if it is post-(semi-)invertible. That is, if there exists an arrow r such that $s \cdot r = \mathrm{id}$.

The dual notion is that of:

Definition 2.2.0.4 (split epimorphism) An arrow r is a **split epimorphism** (or "split epic") if it is pre-(semi-)invertible. That is, if there exists an arrow s such that $s \cdot r = \mathrm{id}$.

It would be perverse to name them this way unless split monics were monic and split epics were epic, which indeed they are.

Lemma 2.2.0.5 A split monomorphism is a monomorphism (and a split epimorphism is an epimorphism).

Proof. Suppose s is split-monic with $s \cdot r = \mathrm{id}$,

$$
\begin{array}{ll}
 & f \cdot s = g \cdot s \\
\Longrightarrow & [\text{whiskering}] \\
 & f \cdot s \cdot r = g \cdot s \cdot r \\
\Longrightarrow & [\text{assumption}] \\
 & f \cdot \mathrm{id} = g \cdot \mathrm{id} \\
\Longrightarrow & [\text{composition unit law}] \\
 & f = g
\end{array}
$$

The other case is dual. □

Because *functors* must preserve the composition structure of categories, they must preserve split monics and epics as well.

Lemma 2.2.0.6 The functor-image of a split monic (split epic) is itself split monic (split epic).

Proof. Suppose s is split-monic with $s \cdot r = \mathrm{id}$,

$$\begin{array}{rl} & \mathrm{F}(s) \cdot \mathrm{F}(r) \\ = & [\text{functors preserve composition}] \\ & \mathrm{F}(s \cdot r) \\ = & [\text{assumption}] \\ & \mathrm{F}(\mathrm{id}) \\ = & [\text{functors preserve identities}] \\ & \mathrm{id} \end{array}$$

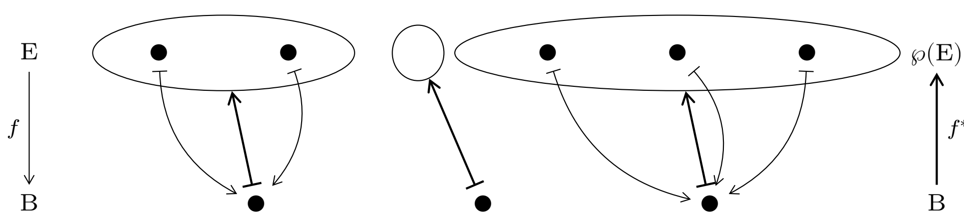
So $\mathrm{F}(s)$ is split-monic. □

Before moving on, let's consider a particularly pretty application of behavioral reasoning to the axiom of choice. This proposition states that given a family of non-empty sets, there is a function that chooses an element from each one.

We can represent any family of sets with an ordinary function in the following way. Given a function $f : \textsc{Set}\,(\mathrm{E} \to \mathrm{B})$, we can define a function,

$$\begin{array}{ccl} \mathrm{B} & \xrightarrow{\underline{f^*}} & \wp(\mathrm{E}) \hookrightarrow \textsc{Set} \\ b & \longmapsto & \{e \in \mathrm{E} \mid f(e) = b\} \end{array}$$

And given a family of sets, $\{\mathrm{E}_b\}_{b \in \mathrm{B}}$, which is just an function $\mathrm{B} \longrightarrow \textsc{Set}$, we can define a function $\int_{b \in \mathrm{B}} \mathrm{E}_b \longrightarrow \mathrm{B}$ mapping $e \in \mathrm{E}_b \longmapsto b$. These two constructions are inverse, they both just sort the elements of E by those in B:



The **axiom of choice** states that if for each $b \in \mathrm{B}$ the set $f^*(b)$ is non-empty then there is a way to choose from E a family of elements $\{e_b\}_{b \in \mathrm{B}}$ such that $\forall b \in \mathrm{B}\ .\ f(e_b) = b$ – i.e. such that there is a function $s : \mathrm{B} \longrightarrow \mathrm{E}$ with $s \cdot f = \mathrm{id}(\mathrm{B})$. Notice that the condition that the sets $f^*(b)$ be non-empty is equivalent to the requirement that f be a surjection. So the axiom of choice asserts that in the category Set, every epimorphism is split!

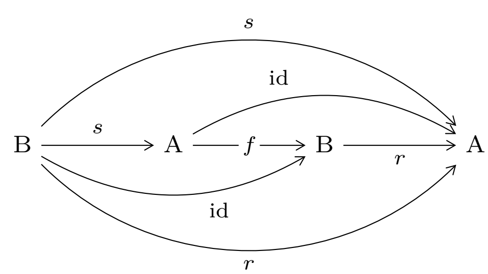
This is a behavioral characterization of a property that we may ask whether a given category satisfies. For example, because the inclusion $(\mathbb{N},+,0) \hookrightarrow (\mathbb{Z},+,0)$ is epic in the category Mon, it fails to hold there.

2.3 Isomorphisms

When we have arrows $s : \mathrm{A} \longrightarrow \mathrm{B}$ and $r : \mathrm{B} \longrightarrow \mathrm{A}$ such that $s \cdot r = \mathrm{id}(\mathrm{A})$ we say that s is a **section** of r and that r is a **retraction** of s. So *being* split monic means *having* a retraction, and *being* split epic means *having* a section.

Lemma 2.3.0.7 If a morphism has both a section and a retraction then the section and the retraction are identical.

Proof. Given arrow $f : \mathrm{A} \longrightarrow \mathrm{B}$ with section s and retraction r, by pasting, $s = r$:



□

If $f : \mathrm{A} \longrightarrow \mathrm{B}$ has section-and-retraction g, then $g : \mathrm{B} \longrightarrow \mathrm{A}$ necessarily has retraction-and-section f. In other words, f and g are (two-sided) inverses for one another. This leads us to a good behavioral characterization of what it means for objects to be *essentially the same* in any category.

Definition 2.3.0.8 (isomorphism) An arrow $f : \mathrm{A} \longrightarrow \mathrm{B}$ is an **isomorphism** if there exists an *anti-parallel* arrow $g : \mathrm{B} \longrightarrow \mathrm{A}$, called an **inverse** of f, such that:

$$f \cdot g = \mathrm{id}(\mathrm{A}) \qquad \text{and} \qquad g \cdot f = \mathrm{id}(\mathrm{B})$$

It follows from lemma 2.3.0.7 that an inverse of f is unique, so we can write it unambiguously as "f^{-1}". To indicate the existence of an unspecified isomorphism between objects A and B, we write "$\mathrm{A} \cong \mathrm{B}$" and call the objects **isomorphic**.

Isomorphism is the right notion of "essentially the same" for category theory because in categories we must characterize objects behaviorally, and there is often no effective way to distinguish between objects that behave identically.

Exercise 2.3.0.9 Show that isomorphism of objects is an equivalence relation; that is, for any objects A, B and C,

reflexivity $\mathrm{A} \cong \mathrm{A}$

symmetry $\mathrm{A} \cong \mathrm{B}$ implies $\mathrm{B} \cong \mathrm{A}$

transitivity $\mathrm{A} \cong \mathrm{B}$ and $\mathrm{B} \cong \mathrm{C}$ implies $\mathrm{A} \cong \mathrm{C}$

Chapter 3

Universal Constructions

A **universal construction** is a description of a construction within a category that determines it uniquely up to a canonical isomorphism. This is the best kind of description we can hope for in a behavioral setting, where we do not have direct access to the internal structure of the objects we are working with.

Universal constructions are defined using **universal properties** that assert that the construction itself has some property, and that if any other construction has the same property then there is a canonical relationship between the two.

In this chapter we introduce the universal constructions needed for the categorical interpretation of intuitionistic propositional proof theory. We do this in a deliberately methodical way, in order to emphasize the similarities in the constructions.

3.1 Terminal and Initial Objects

3.1.1 Terminal Objects

In the category Set, a singleton set S has the property that given any set X there is a unique function from X to S, namely, the constant function on the only element of S. This is a behavioral characterization that we may state in an arbitrary category.

Definition 3.1.1.1 (terminal object) In any category, a **terminal object** is an object T with the property that for any object X there is a unique morphism $x : \mathrm{X} \longrightarrow \mathrm{T}$.

We write "!(X)" for the unique map from an object X to a terminal object and refer to it as a **bang map**.

Whenever some construction has a certain relationship to all constructions of the same kind within a category, it must, in particular, have this relationship to itself. Socrates' dictum to "know thyself" is as important in category theory as it is in life. Consequently, whenever we encounter a universal construction we will see what we can learn about it by "probing it with itself". In the case of a terminal object, this means choosing $\mathrm{X} := \mathrm{T}$ in the definition.

Lemma 3.1.1.2 (identity expansion for terminals) If T is a terminal object then $!(\mathrm{T}) = \mathrm{id}(\mathrm{T})$.

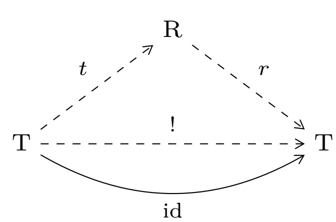
Proof. By definition, $!(\mathrm{T})$ is the unique map $t : \mathrm{T} \longrightarrow \mathrm{T}$, but $\mathrm{id}(\mathrm{T})$ is necessarily in the same hom set. □

Universal constructions are each unique up to a unique structure-preserving isomorphism. In the case of a terminal object, there is no structure to be preserved: it's just a single object. Consequently, we obtain an especially strong uniqueness property.

Lemma 3.1.1.3 (uniqueness of terminals) When they exist, terminal objects are unique up to a unique isomorphism.

Proof. Suppose that T and R are two terminal objects in a category. By assumption, there are unique arrows $t : \mathrm{T} \longrightarrow \mathrm{R}$ and $r : \mathrm{R} \longrightarrow \mathrm{T}$ and:

$$\begin{aligned} & t \cdot r : \mathrm{T} \longrightarrow \mathrm{T} \\ = \quad & [\text{T is terminal}] \\ & !(\mathrm{T}) : \mathrm{T} \longrightarrow \mathrm{T} \\ = \quad & [\text{identity expansion for terminals}] \\ & \mathrm{id}(\mathrm{T}) : \mathrm{T} \longrightarrow \mathrm{T} \end{aligned}$$



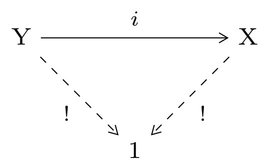
Symmetrically, we have that $r \cdot t = \mathrm{id}(\mathrm{R})$. So t is an *isomorphism*. By the universal property of R, the hom set $\mathrm{T} \longrightarrow \mathrm{R}$ is a singleton, so it must be the only such isomorphism. □

Because terminal objects are unique up to unique isomorphism, we write "1" to refer to an arbitrary terminal object of a category.

Exercise 3.1.1.4 (pre-composing with a bang) Use the universal property of a terminal object to prove the following:

For a terminal object 1 and arrow $i : \mathrm{Y} \longrightarrow \mathrm{X}$,

$$i \cdot !(\mathrm{X}) \quad = \quad !(\mathrm{Y}) \quad : \quad \mathrm{Y} \longrightarrow 1$$



As mentioned, in SET, any singleton set is terminal. Likewise, in Cat, any *singleton category* is. In Mon, the trivial monoid (having only the identity element) is terminal.

Exercise 3.1.1.5 Work out what a terminal object is in the category PreOrd, and determine when a preordered set, as a category, has a terminal object.

3.1.2 Preliminary Interpretation of Truth

The terminal object universal construction provides the categorical interpretation of the logical propositional constant truth,

$$[\![\top]\!] \quad := \quad 1$$

The natural deduction introduction rule for truth,

$$\frac{}{\top}\ \top+$$

is interpreted by the unique map to the terminal object:

$$[\![\top+]\!] \quad := \quad !([\![-]\!])$$

Truth has no elimination rule. We will interpret the rest of the proof theoretic features of truth, and of the other connectives of intuitionistic first-order logic, in a uniform way once we have established some more category theoretic machinery.

3.1.3 Global and Generalized Elements

In Set, there is a bijection between the elements of a set X and the functions from a singleton set to X: to each $x \in \mathrm{X}$ there corresponds the unique function $\ulcorner x \urcorner : 1 \longrightarrow \mathrm{X}$ mapping $\star \longmapsto x$. We can use this behavioral characterization to define an analogue for set membership.

Definition 3.1.3.1 (global element) In a category with a terminal object, a **global element** (or "point") of an object X is an element of the hom set $1 \longrightarrow \mathrm{X}$.

Definition 3.1.3.2 (generalized element) In contrast, a **generalized element** of an object X is just a morphism with codomain X; in other words, an object of the *slice category* over X.

In Set, we can determine whether or not two functions are the same by probing them with points because two parallel functions $f\,,g : \textsc{Set}\,(\mathrm{X} \to \mathrm{Y})$ are *defined* to be the same if $\forall x \in \mathrm{X}\ .\ f(x) = g(x)$. This is the principle of **function extensionality**. Here is a categorical analogue:

Definition 3.1.3.3 (well-pointed category) A category with a terminal object is **well-pointed** if for every $f\,,g : \mathrm{A} \longrightarrow \mathrm{B}$, and global element $a : 1 \longrightarrow \mathrm{A}$,

$$a \cdot f = a \cdot g \qquad \text{implies} \qquad f = g$$

Notice the similarity to the definition of an *epimorphism*. In fact, we can say that the category is well-pointed if its points are *jointly epic*.

Exercise 3.1.3.4 In contrast to the case with global elements, in any category we can determine whether or not two parallel arrows are the same by probing them with *generalized elements*. Prove this.

3.1.4 Initial Objects

The concept *dual* to that of a terminal object is of an initial object.

Definition 3.1.4.1 (initial object) In any category, an **initial object** is an object S with the property that for any object X there is a unique morphism $x : \mathrm{S} \longrightarrow \mathrm{X}$

We write "¡(X)" for the unique map in $\mathrm{S} \longrightarrow \mathrm{X}$ and refer to it as a **cobang map**. By probing an initial object with itself we obtain a result dual to lemma 3.1.1.2:

Lemma 3.1.4.2 (identity expansion for initials) If S is an initial object then ¡(S) = id(S).

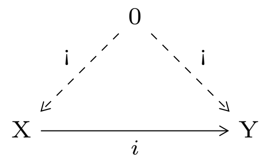
Exercise 3.1.4.3 (uniqueness of initials) Check that when they exist, initial objects are unique up to a unique isomorphism.

We write "0" to refer to an arbitrary initial object of a category.

Lemma 3.1.4.4 (post-composing with a cobang) dual to exercise 3.1.1.4:

For an initial object 0 and arrow $i : \mathrm{X} \longrightarrow \mathrm{Y}$,

$$¡(\mathrm{X}) \cdot i \quad = \quad ¡(\mathrm{Y}) \quad : \quad 0 \longrightarrow \mathrm{Y}$$



In SET, the empty set is initial. Likewise, in CAT, the *empty category* is. In MON, the trivial monoid is initial as well as terminal. (An object which is both terminal and initial is known as a **null object**.)

Exercise 3.1.4.5 Dualize exercise 3.1.1.5 by working out what an initial object is in the category PREORD, and determine when a preordered set, as a category, has an initial object.

3.1.5 Preliminary Interpretation of Falsehood

The initial object universal construction provides the categorical interpretation of the logical propositional constant falsehood,

$$[\![\bot]\!] \quad := \quad 0$$

The natural deduction elimination rule for falsehood,

$$\frac{\bot}{\mathrm{A}} \ \bot\text{–}^{\dagger}$$

is interpreted by the unique map from the initial object[1]:

$$[\![\bot\text{–}^{\dagger}]\!] \quad := \quad ¡([\![\text{–}]\!])$$

Dual to truth, there is no introduction rule.

3.2 Products

3.2.1 Products of Objects

The cartesian product of two sets is defined to be the set of their ordered pairs:

$$\mathrm{A} \times \mathrm{B} \quad := \quad \{(a\,,b) \mid a \in \mathrm{A} \text{ and } b \in \mathrm{B}\}$$

This gives us two projection functions,

$$\pi_0 : \mathrm{A} \times \mathrm{B} \longrightarrow \mathrm{A} \qquad \text{and} \qquad \pi_1 : \mathrm{A} \times \mathrm{B} \longrightarrow \mathrm{B}$$

[1] Actually, we are brushing something under the rug here having to do with contexts. But since we haven't talked about those yet, we'll just leave it lurking there for now, and revisit the issue later.

such that

$$\pi_0(a\,,b) \;=\; a \qquad \text{and} \qquad \pi_1(a\,,b) \;=\; b$$

and further, for any $c \in \mathrm{A} \times \mathrm{B}$,

$$c \;=\; (\pi_0 c\,,\pi_1 c)$$

So having a pair of elements, one from the set A and one from the set B, is the same thing as having a single element of the set $\mathrm{A} \times \mathrm{B}$: given an $a \in \mathrm{A}$ and $b \in \mathrm{B}$ we make an element of $\mathrm{A} \times \mathrm{B}$ by forming the tuple $(a\,,b)$, and given an element $c \in \mathrm{A} \times \mathrm{B}$ we recover an A and a B by taking the projections.

Because not every category is *well-pointed* – or for that matter, even has a terminal object – to obtain a behavioral characterization of a product we need to generalize this description using, appropriately, *generalized elements*.

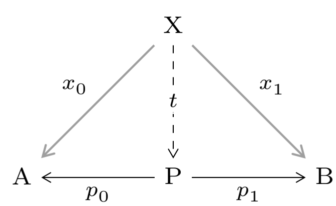
Definition 3.2.1.1 (product of objects) In any category, a (cartesian) **product** of objects A and B is a *span* on A and B,

$$\mathrm{A} \xleftarrow{p_0} \mathrm{P} \xrightarrow{p_1} \mathrm{B}$$

with the property that for any span on A and B,

$$\mathrm{A} \xleftarrow{x_0} \mathrm{X} \xrightarrow{x_1} \mathrm{B}$$

there is a unique map $t : \mathrm{X} \longrightarrow \mathrm{P}$ such that $t \cdot p_0 = x_0$ and $t \cdot p_1 = x_1$:

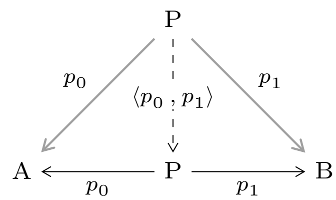


This says that there is a bijection between ordered pairs of maps $(x_0\,,x_1)$ and single maps t such that the diagram commutes. We call A and B the **factors** of the product, p_0 and p_1 its (coordinate) **projections** and t the **tuple** of x_0 and x_1 and write it as "$\langle x_0\,,x_1\rangle$".

Let's see what we can learn by probing a product with itself by choosing $\mathrm{X} := \mathrm{P}$ and $(x_0\,,x_1) := (p_0\,,p_1)$.

Lemma 3.2.1.2 (identity expansion for products) If P is a product of A and B with projections p_0 and p_1, then $\langle p_0\,,p_1\rangle = \mathrm{id}(\mathrm{P})$.

Proof. By assumption, $\langle p_0\,,p_1\rangle$ is the unique map $t : \mathrm{P} \longrightarrow \mathrm{P}$ with the property that, $t \cdot p_0 = p_0$ and $t \cdot p_1 = p_1$:



but by the left unit law of composition, id(P) has this property. □

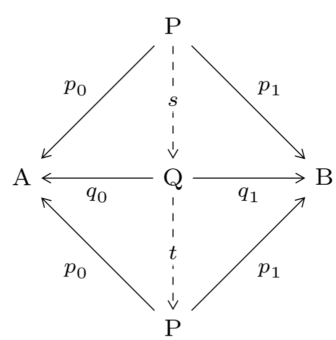
Because products are structures characterized by a universal property, we expect them to be uniquely determined up to a unique structure-preserving isomorphism. This is indeed the case:

Lemma 3.2.1.3 (uniqueness of products) When they exist, products of objects are unique up to a unique projection-preserving isomorphism.

Proof. Suppose that the spans:

$$\mathrm{A} \xleftarrow{p_0} \mathrm{P} \xrightarrow{p_1} \mathrm{B} \qquad \text{and} \qquad \mathrm{A} \xleftarrow{q_0} \mathrm{Q} \xrightarrow{q_1} \mathrm{B}$$

are both products of A and B. Because Q is a product there is a unique $s : \mathrm{P} \longrightarrow \mathrm{Q}$ such that $s \cdot q_0 = p_0$ and $s \cdot q_1 = p_1$. Likewise, because P is a product there is a unique $t : \mathrm{Q} \longrightarrow \mathrm{P}$ such that $t \cdot p_0 = q_0$ and $t \cdot p_1 = q_1$:



Then for $i \in \{0 , 1\}$:

$$\begin{array}{ll} & s \cdot t \cdot p_i \\ = & [\text{P is a product}] \\ & s \cdot q_i \\ = & [\text{Q is a product}] \\ & p_i \end{array}$$

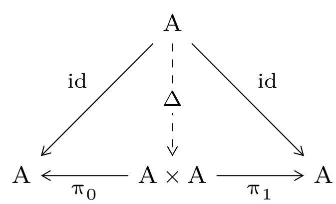
Thus $s \cdot t = \langle p_0 , p_1 \rangle : \mathrm{P} \longrightarrow \mathrm{P}$. By identity expansion for products, $s \cdot t = \mathrm{id}(\mathrm{P})$. Reversing the roles of P and Q, we get that $t \cdot s = \mathrm{id}(\mathrm{Q})$ as well. So s is an isomorphism. By the universal property of Q, it is the only one that respects the coordinate projections. □

Because products are determined as uniquely as is possible by a behavioral characterization, we write "$\mathrm{A} \times \mathrm{B}$" to refer to an arbitrary product of A and B. When the product in question is clear from context, we refer to the two coordinate projections generically as "π_0" and "π_1".

Note that unlike the case with terminal objects, there is not necessarily a unique isomorphism between two products of the same factors. For example, in Set the identity function, $(x , y) \longmapsto (x , y)$, and swap map, $(x , y) \longmapsto (y , x)$, are both isomorphisms $\mathrm{A} \times \mathrm{A} \longrightarrow \mathrm{A} \times \mathrm{A}$. But only the former respects the coordinate projections.

Definition 3.2.1.4 (diagonal map) For every object A, the universal property of the product gives a canonical **diagonal map**, which duplicates its argument:

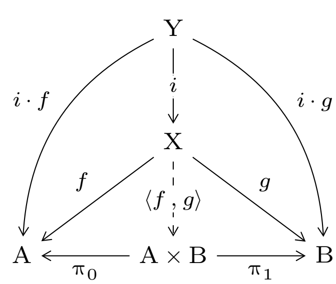
$$\Delta(\mathrm{A}) \quad := \quad \langle \mathrm{id}(\mathrm{A}) , \mathrm{id}(\mathrm{A}) \rangle \quad : \quad \mathrm{A} \longrightarrow \mathrm{A} \times \mathrm{A}$$



Exercise 3.2.1.5 (pre-composing with a tuple) Use the diagram below and the universal property of a product of objects to prove the following:

For a product $\mathrm{A} \times \mathrm{B}$, a tuple $\langle f , g \rangle : \mathrm{X} \longrightarrow \mathrm{A} \times \mathrm{B}$ and an arrow $i : \mathrm{Y} \longrightarrow \mathrm{X}$,

$$i \cdot \langle f , g \rangle \quad = \quad \langle i \cdot f , i \cdot g \rangle \quad : \quad \mathrm{Y} \longrightarrow \mathrm{A} \times \mathrm{B}$$



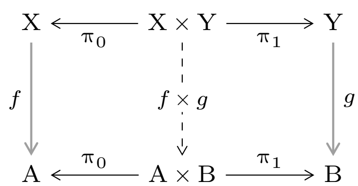
3.2.2 Product Functors

We can use the universal property of a product of objects to define a product of arrows as well:

Definition 3.2.2.1 (product of arrows) Given a pair of arrows $f : \mathrm{X} \longrightarrow \mathrm{A}$ and $g : \mathrm{Y} \longrightarrow \mathrm{B}$, and cartesian products $\mathrm{X} \times \mathrm{Y}$ and $\mathrm{A} \times \mathrm{B}$, we define the **product of arrows** by:

$$\begin{array}{rcl} f \times g & : & \mathrm{X} \times \mathrm{Y} \longrightarrow \mathrm{A} \times \mathrm{B} \\ f \times g & := & \langle \pi_0 \cdot f \,, \pi_1 \cdot g \rangle \end{array}$$

That is, $f \times g$ is the unique arrow making the two squares commute:



This allows us to characterize the cartesian product as a functor:

Lemma 3.2.2.2 (functoriality of products) If a category $\mathbb{C}$ has products for each pair of objects, then the given definition of products for arrows yields a *functor*,

$$- \times - \;:\; \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$$

called the **product functor**.

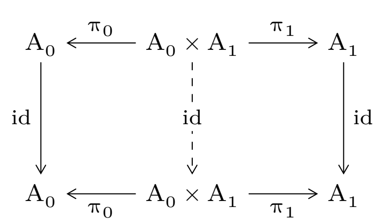
Before giving the proof, we pause to explain this statement, as it is easy to be confused about what is being asserted. In the theorem, "$\mathbb{C} \times \mathbb{C}$" is the *product category* (definition 1.6.1.1). This is indeed a product in the category Cat (as you should check). In contrast, "$- \times -$" is the *name* of an alleged functor having as domain the category $\mathbb{C} \times \mathbb{C}$ and as codomain the category $\mathbb{C}$.

Proof. In order to prove that $- \times -$ is a functor, we must show that it preserves the composition structure.

nullary composition We must show that

$$\mathrm{id}(\mathrm{A}_0) \times \mathrm{id}(\mathrm{A}_1) \;=\; \mathrm{id}(\mathrm{A}_0 \times \mathrm{A}_1)$$

In the diagram,

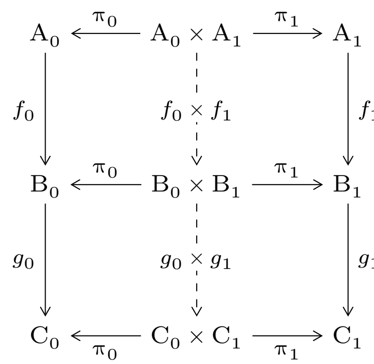


the arrow $\mathrm{id}(A_0 \times A_1)$ makes both squares commute, so the result follows by the definition of product of arrows.

binary composition We must show that

$$(f_0 \cdot g_0) \times (f_1 \cdot g_1) \;=\; (f_0 \times f_1) \cdot (g_0 \times g_1)$$

In the diagram,



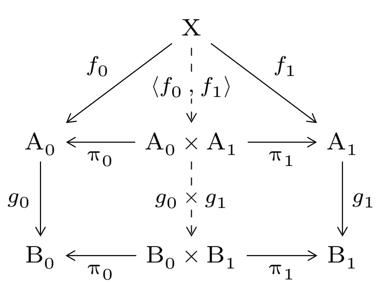
the top two squares commute by the definition of $f_0 \times f_1$ and the bottom two squares commute by the definition of $g_0 \times g_1$. By pasting, the rectangle comprising the two left squares commutes, and likewise the rectangle comprising the two right squares. By definition, $(f_0 \cdot g_0) \times (f_1 \cdot g_1)$ is the unique arrow from $A_0 \times A_1$ to $C_0 \times C_1$ making the outer square commute.

□

Exercise 3.2.2.3 (post-composing a product of arrows) Use the universal property of a product of objects to prove the following:

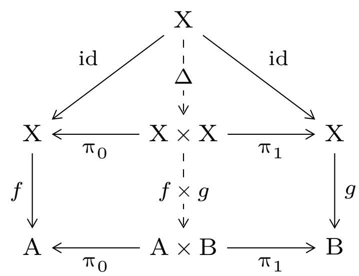
For arrows $\langle f_0 , f_1 \rangle : X \longrightarrow A_0 \times A_1$ and $g_0 \times g_1 : A_0 \times A_1 \longrightarrow B_0 \times B_1$,

$$\langle f_0 , f_1 \rangle \cdot (g_0 \times g_1) \;=\; \langle f_0 \cdot g_0 , f_1 \cdot g_1 \rangle$$



Corollary 3.2.2.4 (tuple factorization) A tuple $\langle f , g \rangle : X \longrightarrow A \times B$ factors through the diagonal as,

$$\langle f , g \rangle \;=\; \Delta(X) \cdot (f \times g)$$



3.2.3 Preliminary Interpretation of Conjunction

The product universal construction provides the categorical interpretation of the logical propositional connective conjunction,

$$[\![A \wedge B]\!] \quad := \quad [\![A]\!] \times [\![B]\!]$$

The natural deduction introduction rule for conjunction,

$$\frac{A \qquad B}{A \wedge B} \;\wedge+$$

which we will write as follows to emphasize that A and B are being proved from the same assumptions,

$$\frac{\Gamma \qquad \dfrac{\dfrac{\overline{\Gamma}}{\mathcal{D}_0}}{A} \qquad \dfrac{\dfrac{\overline{\Gamma}}{\mathcal{D}_1}}{B}}{A \wedge B} \;\wedge+$$

is interpreted by the tuple construction:

$$[\![\wedge+]\!] \quad := \quad \langle [\![-]\!] \,, [\![-]\!] \rangle$$

and the natural deduction elimination rules for conjunction,

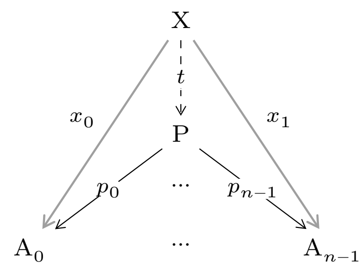
$$\frac{A \wedge B}{A} \;\wedge-_0 \qquad \frac{A \wedge B}{B} \;\wedge-_1$$

are interpreted by the coordinate projections:

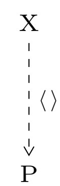
$$[\![\wedge-_0]\!] \quad := \quad \pi_0 \qquad \text{and} \qquad [\![\wedge-_1]\!] \quad := \quad \pi_1$$

3.2.4 Finite Products

Returning to our theme of *unbiased* presentations, we would like to define an n-ary product for each $n \in \mathbb{N}$. Let's think about what the universal property of such a construction would be. A product of n factors would consist of an object P, together with a coordinate projection, $p_i : \mathrm{P} \longrightarrow \mathrm{A}_i$ for each factor such that for any n-ary span $x_i : \mathrm{X} \longrightarrow \mathrm{A}_i$ over the same factors there is a unique n-tuple map $t : \mathrm{X} \longrightarrow \mathrm{P}$ with $t \cdot p_i = x_i$.

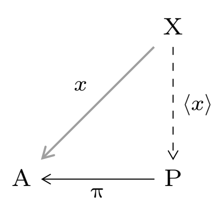


For $n := 0$, a **nullary product** is an object P (requiring no coordinate projections) such that for any object X (requiring no maps to the zero factors) there is a unique null-tuple $\langle\,\rangle : \mathrm{X} \longrightarrow \mathrm{P}$ (satisfying no conditions):



But this is just a *terminal object*!

For $n := 1$, a **unary product** of an object A is an object P with a single coordinate projection, $\pi : \mathrm{P} \longrightarrow \mathrm{A}$ such that for any arrow $x : \mathrm{X} \longrightarrow \mathrm{A}$ there is a unique one-tuple $\langle x \rangle : \mathrm{X} \longrightarrow \mathrm{P}$ with $\langle x \rangle \cdot \pi = x$:



A moment's thought confirms that the choice of $\mathrm{P} := \mathrm{A}$ and $\pi := \mathrm{id}(\mathrm{A})$ (and thus $\langle x \rangle := x$) satisfies this property. So any object is a unary product of itself.

Binary products have already been defined, so we have left to consider products of three or more factors. A ternary product is an object $\mathrm{A} \times \mathrm{B} \times \mathrm{C}$, equipped with three coordinate projection maps such that for any 3-legged span over its factors there is a unique map from the apex to $\mathrm{A} \times \mathrm{B} \times \mathrm{C}$ commuting with the coordinate projections. But this is the same universal property enjoyed by $(A \times B) \times C$, which has projections $\pi_0 \cdot \pi_0$ to A, $\pi_0 \cdot \pi_1$ to B and π_1 to C. Any span over A, B and C contains a subspan over A and B, so by the universal property of $A \times B$, has a unique map from the apex to this product, which together with the C leg of the span gives us a unique map from the apex to $(A \times B) \times C$. The product of four or more factors is analogous.

Of course, there is nothing special about the choice of bracketing:

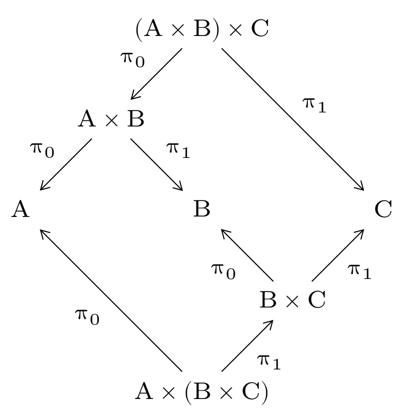
Lemma 3.2.4.1 The cartesian product is associative, up to isomorphism:

$$A \times (B \times C) \;\cong\; (A \times B) \times C$$

Proof. The maps back and forth,

$$s : A \times (B \times C) \longrightarrow (A \times B) \times C \quad \text{and} \quad t : (A \times B) \times C \longrightarrow A \times (B \times C)$$

become clear when we draw the diagram showing how each compound product projects to the three factors, A, B and C:



From this we can simply read off:

$$s \;:=\; \langle\langle \pi_0 \,, \pi_1 \cdot \pi_0 \rangle \,, \pi_1 \cdot \pi_1 \rangle \;:\; A \times (B \times C) \longrightarrow (A \times B) \times C$$

$$t \;:=\; \langle \pi_0 \cdot \pi_0 \,, \langle \pi_0 \cdot \pi_1 \,, \pi_1 \rangle\rangle \;:\; (A \times B) \times C \longrightarrow A \times (B \times C)$$

And then we check:

$$
\begin{array}{rl}
 & s \cdot t \\
= & \text{[definition } t\text{]} \\
 & s \cdot \langle \pi_0 \cdot \pi_0 \,, \langle \pi_0 \cdot \pi_1 \,, \pi_1 \rangle\rangle \\
= & \text{[precomposing with a tuple]} \\
 & \langle s \cdot \pi_0 \cdot \pi_0 \,, \langle s \cdot \pi_0 \cdot \pi_1 \,, s \cdot \pi_1 \rangle\rangle \\
= & \text{[definition } s\text{]} \\
 & \langle \pi_0 \,, \langle \pi_1 \cdot \pi_0 \,, \pi_1 \cdot \pi_1 \rangle\rangle \\
= & \text{[precomposing with a tuple]} \\
 & \langle \pi_0 \,, \pi_1 \cdot \langle \pi_0 \,, \pi_1 \rangle\rangle \\
= & \text{[identity expansion for products]} \\
 & \langle \pi_0 \,, \pi_1 \cdot \mathrm{id} \rangle \\
= & \text{[composition unit law]} \\
 & \langle \pi_0 \,, \pi_1 \rangle \\
= & \text{[identity expansion for products]} \\
 & \mathrm{id}
\end{array}
$$

Similarly, $t \cdot s = \mathrm{id}$. □

Up to isomorphism, the cartesian product has the structure of a monoid:

Lemma 3.2.4.2 A terminal object is a unit for the cartesian product, up to isomorphism:

$$\mathrm{A} \times 1 \;\cong\; \mathrm{A} \;\cong\; 1 \times \mathrm{A}$$

Proof. The projection $\pi_0 : \mathrm{A} \times 1 \longrightarrow \mathrm{A}$ is an isomorphism, with inverse $\langle \mathrm{id}(\mathrm{A}) \,, !(\mathrm{A}) \rangle : \mathrm{A} \longrightarrow \mathrm{A} \times 1$.

- By the universal property of the product,

$$\langle \mathrm{id}(\mathrm{A}) \,, !(\mathrm{A}) \rangle \cdot \pi_0 \;=\; \mathrm{id}(\mathrm{A}) \;:\; \mathrm{A} \longrightarrow \mathrm{A}$$

- Going the other way,

$$
\begin{array}{rl}
 & \pi_0 \cdot \langle \mathrm{id}(\mathrm{A}) \,, !(\mathrm{A}) \rangle \quad : \quad \mathrm{A} \times 1 \longrightarrow \mathrm{A} \times 1 \\
= & \text{[pre-composing with a tuple]} \\
 & \langle \pi_0 \cdot \mathrm{id}(\mathrm{A}) \,, \pi_0 \cdot !(\mathrm{A}) \rangle \\
= & \text{[composition unit law and pre-composing with a bang]} \\
 & \langle \pi_0 \,, !(\mathrm{A} \times 1) \rangle \\
= & \text{[universal property of a terminal object]} \\
 & \langle \pi_0 \,, \pi_1 \rangle \\
= & \text{[identity expansion for products]} \\
 & \mathrm{id}(\mathrm{A} \times 1)
\end{array}
$$

□

And furthermore, this monoid is commutative:

Lemma 3.2.4.3 The cartesian product is symmetric, up to isomorphism:

$$A \times B \;\cong\; B \times A$$

Proof. The **swap map**, $\sigma_{A,B} := \langle \pi_1 , \pi_0 \rangle : A \times B \longrightarrow B \times A$ is an isomorphism, with inverse the swap map, $\sigma_{B,A} := \langle \pi_1 , \pi_0 \rangle : B \times A \longrightarrow A \times B$:

$$\begin{array}{ll} & \sigma_{A,B} \cdot \sigma_{B,A} \\ = & [\text{definition}] \\ & \langle \pi_1 , \pi_0 \rangle \cdot \langle \pi_1 , \pi_0 \rangle \\ = & [\text{pre-composing with a tuple}] \\ & \langle \langle \pi_1 , \pi_0 \rangle \cdot \pi_1 , \langle \pi_1 , \pi_0 \rangle \cdot \pi_0 \rangle \\ = & [\text{universal property of a product}] \\ & \langle \pi_0 , \pi_1 \rangle \\ = & [\text{identity expansion for products}] \\ & \mathrm{id}(A \times B) \end{array}$$

and symmetrically $\sigma_{B,A} \cdot \sigma_{A,B} = \mathrm{id}(B \times A)$ □

To have **finite products** – that is, n-ary products for all $n \in \mathbb{N}$, it suffices to have binary products and a terminal object. A category with all finite products is called a **cartesian category**.

3.2.5 Preliminary Interpretation of Contexts

Finite products provide the categorical structure needed to interpret (non-dependent, structural) contexts. We interpret the empty context with a terminal object:

$$[\![\emptyset]\!] \quad := \quad 1$$

And we interpret context extension with a cartesian product:

$$[\![\Gamma , A]\!] \quad := \quad [\![\Gamma]\!] \times [\![A]\!]$$

Finite products give us just the right structure to inplement what are commonly called the "structural rules" of proof theory. These are most easily described using a *local*, or *sequent*, presentation, in which contexts are recorded explicitly in each inference rule. In such a presentation, a sequent is interpreted as a hom set:

$$[\![\Gamma \;\vdash\; A]\!] \quad := \quad [\![\Gamma]\!] \longrightarrow [\![A]\!]$$

and an inference rule is interpreted as a function that builds a member of the hom set in the conclusion, given a member of each hom set in the premises. For example, the introduction (or right) rule for conjunction,

$$\frac{\Gamma \;\vdash\; A \qquad \Gamma \;\vdash\; B}{\Gamma \;\vdash\; A \wedge B} \;\wedge R$$

is interpreted by the tupling operation, which takes arrows $f : [\![\Gamma \ \vdash \ \mathrm{A}]\!]$ and $g : [\![\Gamma \ \vdash \ \mathrm{B}]\!]$ and returns $\langle f , g \rangle : [\![\Gamma \ \vdash \ \mathrm{A} \wedge \mathrm{B}]\!]$.

The structural rules of context weakening, contraction and exchange specify the properties of contexts. Because contexts are interpreted as the domains of arrows, we expect their interpretations to behave contravariantly.

The rule of **context weakening** says,

$$\frac{\Gamma \ \vdash \ \mathrm{B}}{\Gamma , \mathrm{A} \ \vdash \ \mathrm{B}} \ \textit{cw}$$

Its categorical interpretation must be some means of constructing a member of $[\![\Gamma]\!] \times [\![\mathrm{A}]\!] \longrightarrow [\![\mathrm{B}]\!]$ from a member of $[\![\Gamma]\!] \longrightarrow [\![\mathrm{B}]\!]$. We can do this by simply pre-composing the projection $\pi_0 : [\![\Gamma]\!] \times [\![\mathrm{A}]\!] \longrightarrow [\![\Gamma]\!]$, or equivalently, (up to the unit isomorphism for products) the product of $\mathrm{id}([\![\Gamma]\!])$ and the *bang map* $!([\![\mathrm{A}]\!]) : [\![\mathrm{A}]\!] \longrightarrow [\![\emptyset]\!]$:

$$[\![\Gamma]\!] \times [\![\mathrm{A}]\!] \overset{\mathrm{id} \times !}{\dashrightarrow} [\![\Gamma]\!] \longrightarrow [\![\mathrm{B}]\!]$$

The rule of **context contraction** says,

$$\frac{\Gamma , \mathrm{A} , \mathrm{A} \ \vdash \ \mathrm{B}}{\Gamma , \mathrm{A} \ \vdash \ \mathrm{B}} \ \textit{cc}$$

Its categorical interpretation must be some means of constructing a member of $[\![\Gamma]\!] \times [\![\mathrm{A}]\!] \longrightarrow [\![\mathrm{B}]\!]$ from a member of $[\![\Gamma]\!] \times [\![\mathrm{A}]\!] \times [\![\mathrm{A}]\!] \longrightarrow [\![\mathrm{B}]\!]$. We can do this (up to the associativity isomorphism for products) by simply pre-composing the product of $\mathrm{id}([\![\Gamma]\!])$ and the *diagonal map* $\Delta([\![\mathrm{A}]\!]) : [\![\mathrm{A}]\!] \longrightarrow [\![\mathrm{A}]\!] \times [\![\mathrm{A}]\!]$:

$$[\![\Gamma]\!] \times [\![\mathrm{A}]\!] \overset{\mathrm{id} \times \Delta}{\dashrightarrow} [\![\Gamma]\!] \times [\![\mathrm{A}]\!] \times [\![\mathrm{A}]\!] \longrightarrow [\![\mathrm{B}]\!]$$

The rule of **context exchange** says,

$$\frac{\Gamma , \mathrm{B} , \mathrm{A} \ \vdash \ \mathrm{C}}{\Gamma , \mathrm{A} , \mathrm{B} \ \vdash \ \mathrm{C}} \ \textit{cx}$$

Its categorical interpretation must be some means of constructing a member of $[\![\Gamma]\!] \times [\![\mathrm{A}]\!] \times [\![\mathrm{B}]\!] \longrightarrow [\![\mathrm{C}]\!]$ from a member of $[\![\Gamma]\!] \times [\![\mathrm{B}]\!] \times [\![\mathrm{A}]\!] \longrightarrow [\![\mathrm{C}]\!]$. We can do this (up to the associativity isomorphism for products) by simply pre-composing the product of $\mathrm{id}([\![\Gamma]\!])$ and the *swap map* $\sigma([\![\mathrm{A}]\!], [\![\mathrm{B}]\!]) : [\![\mathrm{A}]\!] \times [\![\mathrm{B}]\!] \longrightarrow [\![\mathrm{B}]\!] \times [\![\mathrm{A}]\!]$:

$$[\![\Gamma]\!] \times [\![\mathrm{A}]\!] \times [\![\mathrm{B}]\!] \overset{\mathrm{id} \times \sigma}{\dashrightarrow} [\![\Gamma]\!] \times [\![\mathrm{B}]\!] \times [\![\mathrm{A}]\!] \longrightarrow [\![\mathrm{C}]\!]$$

So a system that a proof theorist might call "structural", a category theorist would call "cartesian": it is one in which we may freely duplicate and delete (as well as swap the order of) the assumptions in our contexts, without the need for explicit justification.

Exercise 3.2.5.1 The **initial sequent** and **cut** structural rules, say, respectively,

$$\frac{}{\Gamma\,,\mathrm{A}\ \vdash\ \mathrm{A}}\ \mathit{init} \qquad \text{and} \qquad \frac{\Gamma\ \vdash\ \mathrm{A} \qquad \Gamma\,,\mathrm{A}\ \vdash\ \mathrm{B}}{\Gamma\ \vdash\ \mathrm{B}}\ \mathit{cut}$$

Use the structure of a cartesian category to define interpretations for these rules.

3.3 Coproducts

A coproduct is the dual construction to a product. Categorically, that is all there is to say about the matter. But because of the asymmetry of proof-theoretic derivations – proceeding from a *collection* of assumptions, interpreted conjunctively, to a *single* conclusion – we will have to say a bit more when it comes to our categorical semantics for proof theory.

First, we record for convenience, but without further comment, the duals of our main results about products. If you're new to all this, it would be an excellent exercise first to go back and see why these are the respective dual theorems, and then to prove each one explicitly – that is, by actually going through the argument, rather than by just saying, "by duality, Qed".

3.3.1 Coproducts of Objects

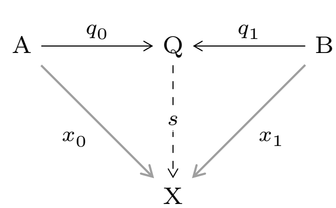
Definition 3.3.1.1 (coproduct of objects) In any category, a **coproduct** of objects A and B is a *cospan* on A and B,

$$\mathrm{A} \xrightarrow{q_0} \mathrm{Q} \xleftarrow{q_1} \mathrm{B}$$

with the property that for any cospan on A and B,

$$\mathrm{A} \xrightarrow{x_0} \mathrm{X} \xleftarrow{x_1} \mathrm{B}$$

there is a unique map $s : \mathrm{Q} \longrightarrow \mathrm{X}$ such that $q_0 \cdot s = x_0$ and $q_1 \cdot s = x_1$:



We call A and B the **cases** of the coproduct, q_0 and q_1 its **insertions** and s the **cotuple** of x_0 and x_1, and write it as "$[x_0\ , x_1]$".

Probing a coproduct with itself by choosing $X := Q$ and $(x_0\ , x_1) := (q_0\ , q_1)$, we learn:

Lemma 3.3.1.2 (identity expansion for coproducts) If Q is a coproduct of A and B with insertions q_0 and q_1, then $[q_0\ , q_1] = \mathrm{id}(Q)$.

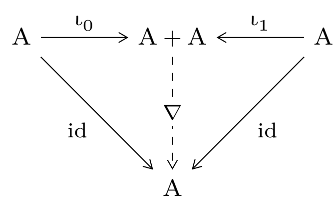
And being characterized by a universal property, we expect:

Lemma 3.3.1.3 (uniqueness of coproducts) When they exist, coproducts of objects are unique up to a unique insertion-preserving isomorphism.

We write "A + B" to refer to an arbitrary coproduct of A and B, When the coproduct in question is clear from context, we refer to the two case insertions generically as "ι_0" and "ι_1".

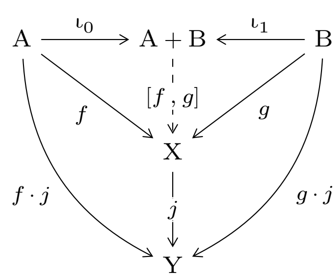
Definition 3.3.1.4 (codiagonal map) For every object A, the universal property of the coproduct gives a canonical **codiagonal map**, which forgets about case distinction:

$$\nabla(A) \quad := \quad [\mathrm{id}(A)\ , \mathrm{id}(A)] \quad : \quad A + A \longrightarrow A$$



Lemma 3.3.1.5 (post-composing with a cotuple) For a coproduct A + B, a cotuple $[f\ , g] : A + B \longrightarrow X$ and an arrow $j : X \longrightarrow Y$,

$$[f\ , g] \cdot j \;\; = \;\; [f \cdot j\ , g \cdot j] \;\; : \;\; A + B \longrightarrow Y$$



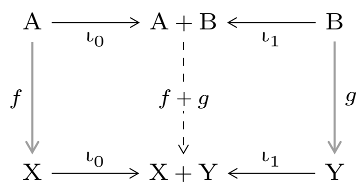
3.3.2 Coproduct Functors

Definition 3.3.2.1 (coproduct of arrows) Given a pair of arrows $f : A \longrightarrow X$ and $g : B \longrightarrow Y$, and coproducts $A + B$ and $X + Y$, we define the **coproduct of**

arrows $f + g : \mathrm{A} + \mathrm{B} \longrightarrow \mathrm{X} + \mathrm{Y}$ by:

$$\begin{array}{rcl} f + g & : & \mathrm{A} + \mathrm{B} \longrightarrow \mathrm{X} + \mathrm{Y} \\ f + g & := & [f \cdot \iota_0 \,, g \cdot \iota_1] \end{array}$$

That is, $f + g$ is the unique arrow making the two squares commute:



This allows us to characterize the coproduct as a functor:

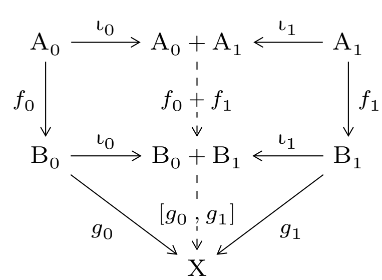
Lemma 3.3.2.2 (functoriality of coproducts) If a category $\mathbb{C}$ has coproducts for each pair of objects, then the given definition of coproducts for arrows yields a *functor*,

$$- + - \;:\; \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$$

called the **coproduct functor**.

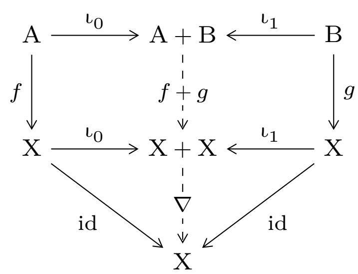
Lemma 3.3.2.3 (pre-composing a coproduct of arrows) For arrows $f_0 + f_1 : \mathrm{A}_0 + \mathrm{A}_1 \longrightarrow \mathrm{B}_0 + \mathrm{B}_1$ and $[g_0 \,, g_1] : \mathrm{B}_0 + \mathrm{B}_1 \longrightarrow \mathrm{X}$,

$$(f_0 + f_1) \cdot [g_0 \,, g_1] \;=\; [f_0 \cdot g_0 \,, f_1 \cdot g_1]$$



Corollary 3.3.2.4 (cotuple factorization) A cotuple $[f \,, g] : \mathrm{A} + \mathrm{B} \longrightarrow \mathrm{X}$ factors through the codiagonal as,

$$[f \,, g] \;=\; (f + g) \cdot \nabla(\mathrm{X})$$



3.3.3 Preliminary Interpretation of Disjunction

The coproduct universal construction provides the categorical interpretation of the logical propositional connective disjunction,

$$[\![\mathrm{A} \vee \mathrm{B}]\!] \quad := \quad [\![\mathrm{A}]\!] + [\![\mathrm{B}]\!]$$

The natural deduction introduction rules for disjunction,

$$\frac{\mathrm{A}}{\mathrm{A} \vee \mathrm{B}}\ \vee{+}_0 \qquad \frac{\mathrm{B}}{\mathrm{A} \vee \mathrm{B}}\ \vee{+}_1$$

are interpreted by the case insertions:

$$[\![\vee{+}_0]\!] \quad := \quad \iota_0 \qquad \text{and} \qquad [\![\vee{+}_1]\!] \quad := \quad \iota_1$$

Using the cotuple construction, we are able to interpret the restricted case of disjunction elimination in which there is no ambient context:

$$\dfrac{\mathrm{A} \vee \mathrm{B} \qquad \dfrac{\dfrac{\overline{\mathrm{A}}}{\mathcal{D}_1}}{\mathrm{C}} \qquad \dfrac{\dfrac{\overline{\mathrm{B}}}{\mathcal{D}_2}}{\mathrm{C}}}{\mathrm{C}}\ \vee{-}^{\dagger}$$

in which case we have:

$$[\![\vee{-}^{\dagger}]\!] := [[\![-]\!]\,,[\![-]\!]]$$

But in order to interpret the full rule for disjunction elimination,

$$\dfrac{\Gamma \qquad \mathrm{A} \vee \mathrm{B} \qquad \dfrac{\dfrac{\overline{\Gamma} \quad \overline{\mathrm{A}}}{\mathcal{D}_1}}{\mathrm{C}} \qquad \dfrac{\dfrac{\overline{\Gamma} \quad \overline{\mathrm{B}}}{\mathcal{D}_2}}{\mathrm{C}}}{\mathrm{C}}\ \vee{-}$$

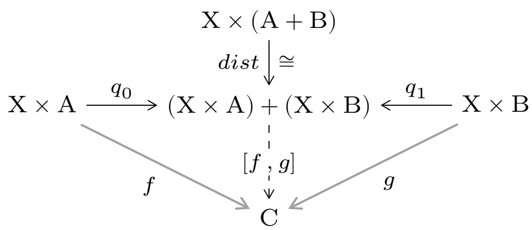
we need a categorical setting in which the **distributive law** holds:

$$\mathit{dist} \quad : \quad \mathrm{X} \times (\mathrm{A} + \mathrm{B}) \;\cong\; (\mathrm{X} \times \mathrm{A}) + (\mathrm{X} \times \mathrm{B})$$

since, in this setting we could define

$$[\![\vee-]\!] \quad := \quad dist \cdot [-\,,-]$$

or



We do not *yet* have a justification for why the distributive law should hold in our (admittedly underspecified) categorical setting, but we will remedy this soon.[2]

3.3.4 Finite Coproducts

Dual to the case for products, we can give an *unbiased* description of finite coproducts, where the **nullary coproduct** is an *initial object*, a **unary coproduct** is just the identity function, and for any $n \geq 3$, the n-ary coproduct is just the iterated binary coproduct bracketed any way you like since,

Lemma 3.3.4.1 The coproduct is associative, up to isomorphism:

$$\mathrm{A} + (\mathrm{B} + \mathrm{C}) \;\cong\; (\mathrm{A} + \mathrm{B}) + \mathrm{C}$$

Furthermore, it forms a monoid:

Lemma 3.3.4.2 An initial object is a unit for the coproduct, up to isomorphism:

$$\mathrm{A} + 0 \;\cong\; \mathrm{A} \;\cong\; 0 + \mathrm{A}$$

And that monoid is commutative:

Lemma 3.3.4.3 The coproduct is symmetric, up to isomorphism:

$$\mathrm{A} + \mathrm{B} \;\cong\; \mathrm{B} + \mathrm{A}$$

To have **finite coproducts** – that is, n-ary coproducts for all $n \in \mathbb{N}$, it suffices to have binary coproducts and an initial object. A category with all finite coproducts is called a **cocartesian category**. A category with all finite products and coproducts is known as **bicartesian**.

[2] If you feel so inclined, you may go back and peak under the rug now.

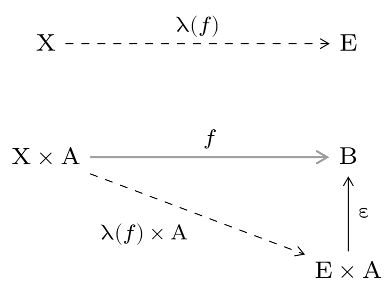
3.4 Exponentials

As functional programmers, we are familiar with the idea of function **currying**, that is, of viewing a function of two arguments as a *higher-order function* that takes the first argument and returns a new function, which, when provided the second argument, computes the same result as the original function when given both arguments at once. Once we get used to working with high-order functions, we wonder how we ever managed to program any other way.

3.4.1 Exponentials of Objects

Exponential objects are the categorical analogue of set-theoretic function space, allowing us to characterize function currying and λ-abstraction.

Definition 3.4.1.1 (exponential object) In a category with binary products, an **exponential** of objects A and B is an object E together with an arrow $\varepsilon : E \times A \longrightarrow B$ with the property that for any object X and arrow $f : X \times A \longrightarrow B$ there is a unique arrow $\lambda(f) : X \longrightarrow E$ such that $\lambda(f) \times A \cdot \varepsilon = f$:



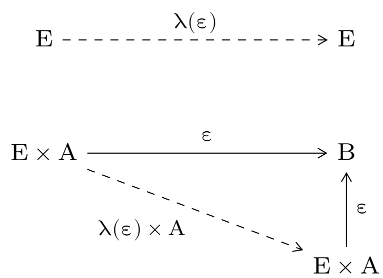
We call ε the **evaluation map** of the exponential, and $\lambda(f)$ the **exponential transpose** or "curry" of f.

Notice that uniqueness together with the "such that" clause lets us recover f from $\lambda(f)$: just take the product with id(A) and compose with ε. This is just "uncurrying" to functional programmers. If this isn't clear to you, go back to the definitions of *product* and *coproduct* and see how the same principle allows us to recover f and g from $\langle f , g \rangle$ and from $[f , g]$, respectively.

Let's see what we learn by probing an exponential with itself by choosing $X := E$ and $f := \varepsilon$.

Lemma 3.4.1.2 (identity expansion for exponentials) If E is an exponential of A and B then $\lambda(\varepsilon) = \mathrm{id}(E)$.

Proof. By assumption, $\lambda(\varepsilon)$ is the unique map in the hom set $E \longrightarrow E$ with the property that $\lambda(\varepsilon) \times A \cdot \varepsilon = \varepsilon$:



By the left unit law of composition, $id(E \times A) \cdot \varepsilon = \varepsilon$, and by the definition of a product of arrows, $id(E \times A) = id(E) \times id(A)$. Since $id(E)$ has the desired property, the result follows from uniqueness. □

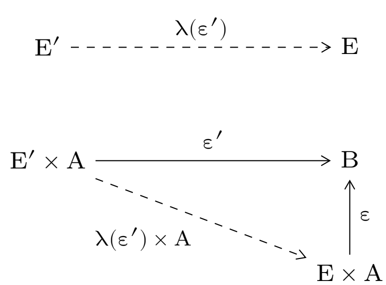
To summarize:

- currying the evaluation map yields the identity on the exponential, and
- uncurrying the identity on the exponential yields the evaluation map.

Because exponentials are structures characterized by a universal property, we expect them to be unique up to a structure-preserving isomorphism. This should be familiar by now.

Lemma 3.4.1.3 (uniqueness of expoentials) When they exist, exponentials are unique up to a unique evaluation-preserving isomorphism.

Proof. Suppose that $(E, \varepsilon, \lambda)$ and $(E', \varepsilon', \lambda')$ are both exponentials of A and B. By setting $X := E'$ and $f := \varepsilon'$ in the universal property of E, we have:

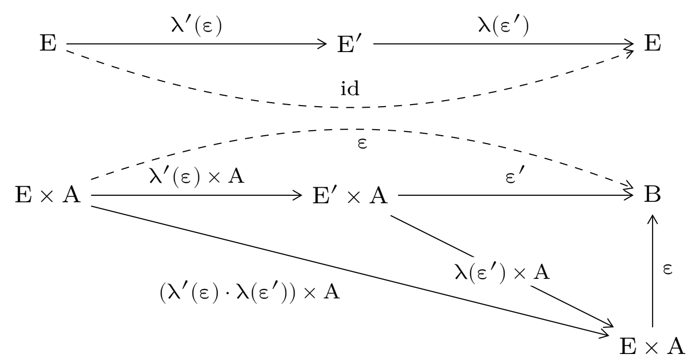


That is, $\lambda(\varepsilon') \times A \cdot \varepsilon = \varepsilon'$. Symmetrically, by the universal property of E', we have $\lambda'(\varepsilon) \times A \cdot \varepsilon' = \varepsilon$.

We want to show that $\lambda'(\varepsilon) \cdot \lambda(\varepsilon') : E \longrightarrow E$ is the identity map. We do so by uncurrying it:

$$
\begin{aligned}
& (\lambda'(e) \cdot \lambda(\varepsilon')) \times \mathrm{id}(A) \cdot \varepsilon \\
= \ & [\text{product functor}] \\
& \lambda'(e) \times \mathrm{id}(A) \cdot \lambda(\varepsilon') \times \mathrm{id}(A) \cdot \varepsilon \\
= \ & [\text{universal property of E}] \\
& \lambda'(e) \times \mathrm{id}(A) \cdot \varepsilon' \\
= \ & [\text{universal property of E}'] \\
& \varepsilon
\end{aligned}
$$

So by identity expansion for exponentials, $\lambda'(\varepsilon) \cdot \lambda(\varepsilon') = \mathrm{id}(E)$.



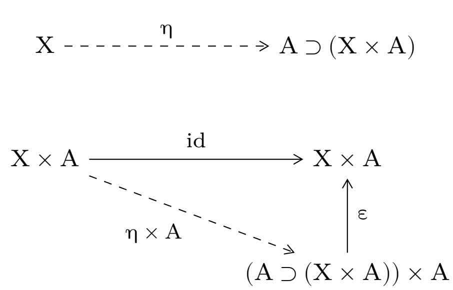
Similarly, we have $\lambda(\varepsilon') \cdot \lambda'(\varepsilon) = \mathrm{id}(E')$.

So $\lambda'(\varepsilon) : E \longrightarrow E'$ is an isomorphism. By the universal property of E′, it is the only one that respects the evaluation ε'. □

Because exponentials are determined as uniquely as is possible by a behavioral characterization, we write "$A \supset B$" to refer to an arbitrary exponential of A and B. The notation "B^A" is also common, and we will use it occasionally, in cases where the pun is a useful mnemonic.

Definition 3.4.1.4 (pairing map) For every object A, the universal property of the exponential gives a canonical **pairing map**, which is a higher-order function that pairs an argument with a given parameter:

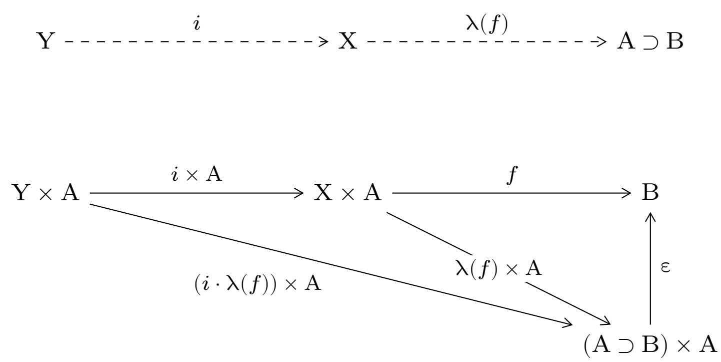
$$\eta(X) \quad := \quad \lambda(\mathrm{id}(X \times A)) \quad : \quad X \longrightarrow A \supset (X \times A)$$



Exercise 3.4.1.5 (pre-composing with a curry) Use the diagram and the universal property of an exponential object to prove the following:

For a exponential $\mathrm{A} \supset \mathrm{B}$, a curry $\lambda f : \mathrm{X} \longrightarrow \mathrm{A} \supset \mathrm{B}$ and an arrow $i : \mathrm{Y} \longrightarrow \mathrm{X}$,

$$i \cdot \lambda f \;=\; \lambda(i \times \mathrm{A} \cdot f) \;:\; \mathrm{Y} \longrightarrow \mathrm{A} \supset \mathrm{B}$$



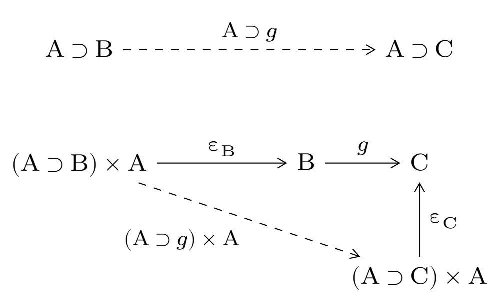
3.4.2 Exponential Functors

We can use the universal property of exponential objects to define a covariant exponential functor.

Definition 3.4.2.1 (covariant exponential of an arrow) For a fixed object A, we define the **exponential of an arrow** $g : \mathrm{B} \longrightarrow \mathrm{C}$ to be:

$$\mathrm{A} \supset g \quad := \quad \lambda(\varepsilon_{\mathrm{B}} \cdot g) \quad : \quad \mathrm{A} \supset \mathrm{B} \longrightarrow \mathrm{A} \supset \mathrm{C}$$

By the universal property of exponentials, $\mathrm{A} \supset g$ is the unique arrow making the triangle commute:



If this definition seems rather unmotivated, it may help to recall that the curry of an evaluation map is the identity (lemma 3.4.1.2) and to keep in mind that the idea behind an exponential of an arrow, $\mathrm{A} \supset g$, is to “post-compose g”.

Lemma 3.4.2.2 (functoriality of exponentials) In a category $\mathbb{C}$ with finite products and a fixed object A, the given definition of exponential of arrows yields a functor,

$$\mathrm{A} \supset - \;:\; \mathbb{C} \longrightarrow \mathbb{C}$$

Proof. In order to prove that $\mathrm{A} \supset -$ is a functor, we must show that it preserves the composition structure.

nullary composition We must show that

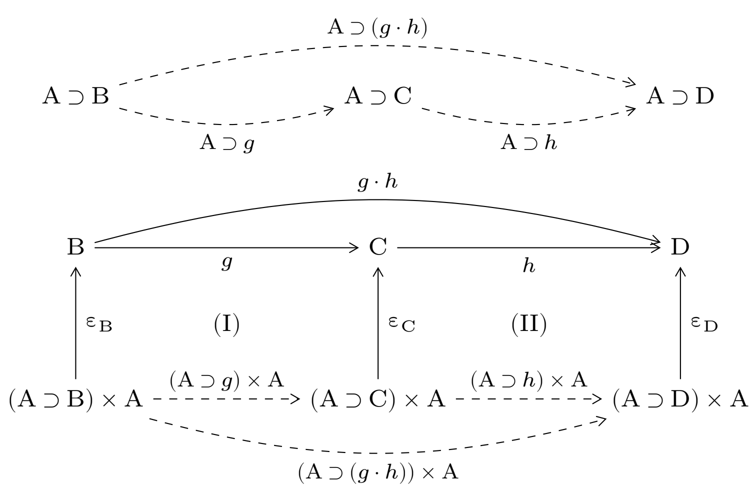
$$\mathrm{A} \supset \mathrm{id}(\mathrm{B}) \;=\; \mathrm{id}(\mathrm{A} \supset \mathrm{B})$$

$$\begin{array}{ll} & \mathrm{A} \supset \mathrm{id}(\mathrm{B}) \\ = & [\text{definition of } \mathrm{A} \supset - \text{ on arrows}] \\ & \lambda(\varepsilon_{\mathrm{B}} \cdot \mathrm{id}(\mathrm{B})) \\ = & [\text{composition unit law}] \\ & \lambda(\varepsilon_{\mathrm{B}}) \\ = & [\text{identity expansion for exponentials}] \\ & \mathrm{id}(\mathrm{A} \supset \mathrm{B}) \end{array}$$

binary composition We must show that

$$\mathrm{A} \supset (g \cdot h) \;=\; \mathrm{A} \supset g \cdot \mathrm{A} \supset h$$

In the diagram,



the outer square commutes by the definition of $\mathrm{A} \supset (g \cdot h)$ and the inner squares (I) and (II) commute by the definitions of $\mathrm{A} \supset g$ and $\mathrm{A} \supset h$, respectively. By the functoriality of the cartesian product,

$$(\mathrm{A} \supset g) \times \mathrm{A} \cdot (\mathrm{A} \supset h) \times \mathrm{A} \quad = \quad (\mathrm{A} \supset g \cdot \mathrm{A} \supset h) \times \mathrm{A}$$

By pasting squares (I) and (II), we see that:

$$\varepsilon_{\mathrm{B}} \cdot (g \cdot h) \quad = \quad (\mathrm{A} \supset g) \times \mathrm{A} \cdot (\mathrm{A} \supset h) \times \mathrm{A} \cdot \varepsilon_{\mathrm{D}}$$

The result follows by the uniqueness clause of the universal property of exponentials.

□

3.4.3 Preliminary Interpretation of Implication

The exponential universal construction provides the categorical interpretation of the logical propositional connective implication,

$$[\![\mathrm{A} \supset \mathrm{B}]\!] \quad := \quad [\![\mathrm{A}]\!] \supset [\![\mathrm{B}]\!]$$

Depending on your point of view, it is an either happy or unhappy coincidence that we have overloaded the symbol “⊃” in this way.

The natural deduction introduction rule for implication,

$$\dfrac{\dfrac{\dfrac{[\mathrm{A}]}{\mathcal{D}}}{\mathrm{B}}}{\mathrm{A} \supset \mathrm{B}} \;\supset+$$

which we will write with explicit context as,

$$\dfrac{\Gamma \qquad \dfrac{\dfrac{\overline{\Gamma} \quad \overline{\mathrm{A}}}{\mathcal{D}}}{\mathrm{B}}}{\mathrm{A} \supset \mathrm{B}} \;\supset+$$

is interpreted by currying:

$$[\![\supset+]\!] \quad := \quad \lambda([\![-]\!])$$

and the natural deduction elimination rule for implication,

$$\dfrac{\mathrm{A} \supset \mathrm{B} \qquad \mathrm{A}}{\mathrm{B}} \;\supset-$$

is interpreted by the evaluation map:

$$[\![\supset-]\!] \quad := \quad \varepsilon$$

3.5 Bicartesian Closed Categories

We have now met all of the universal constructions needed to interpret intuitionistic *propositional* proof theory.

A category having all finite products (i.e. a terminal object and binary products), as well as all exponentials is known as a **cartesian closed category**. A category that is both *bicartesian* and *cartesian closed* is called a **bicartesian closed category**. We will call the subcategory of CAT consisting of bicartesian closed categories, their (bicartesian closed) structure-preserving functors "BCC".

So far, we have introduced the interpretations of the introduction and elimination rules of the propositional connectives in a bicartesian closed category. However, we have not yet introduced interpretations for the rest of the proof-theoretical structure (i.e. local reductions and expansions and permutation conversions).

If we were interested in studying only *provability*, rather than *proofs* themselves, we could stop here. In that case, we could interpret intuitionistic propositional logic in a bicartesian closed preorder (or poset). These structures are well-known in lattice theory, where they are called **Heyting algebras**.

However, because we are interested in proof theory we will ultimately want to interpret intuitionistic propositional logic into *free* bicartesian closed categories – those whose arrows, as well as the equations between them, are exactly the ones required by the bicartesian closed structure, with no "junk" (additional arrows) and no "confusion" (additional equations between them).

3.5.1 Context Distributivity

We still have pending the issue of **context distributivity**. For the propositional connective disjunction, in order to recover the proof theoretic elimination rule from the special case of the cotuple, we need to know that the *distributive law* holds. Likewise, for the propositional constant falsehood, we need nullary context distributivity or **absorption law**,

$$abs \quad : \quad \mathrm{X} \times 0 \;\cong\; 0$$

It turns out that in any bicartesian closed category these isomorphisms hold. We don't have time to give a full explanation of why this is the case, but the short version is that they're consequences of the **Yoneda principle**, which says essentially that for given objects $\mathrm{X}, \mathrm{Y} : \mathbb{C}$, if for all objects Z, we have $\mathbb{C}(\mathrm{X} \to \mathrm{Z}) \cong \mathbb{C}(\mathrm{Y} \to \mathrm{Z})$ in SET then $\mathrm{X} \cong \mathrm{Y}$ in $\mathbb{C}$. You should think of this as generalizing the following statement in a preordered set:

$$(\forall \mathrm{Z} \,.\, \mathrm{X} \leq \mathrm{Z} \Longleftrightarrow \mathrm{Y} \leq \mathrm{Z}) \qquad \text{implies} \qquad \mathrm{X} \quad \cong \quad \mathrm{Y}$$

Using this principle, the (nullary and binary) distributive laws follow from:

$$
\begin{array}{ll}
 & \mathbb{C}\,(\mathrm{X} \times 0 \to \mathrm{Z}) \\
\cong & [\text{product symmetry}] \\
 & \mathbb{C}\,(0 \times \mathrm{X} \to \mathrm{Z}) \\
\cong & [\text{currying}] \\
 & \mathbb{C}\,(0 \to \mathrm{X} \supset \mathrm{Z}) \\
\cong & [\text{universal property of } 0] \\
 & \mathbb{C}\,(0 \to \mathrm{Z})
\end{array}
$$

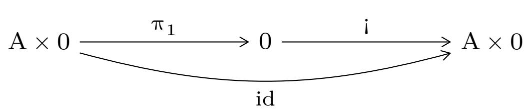
$$
\begin{array}{ll}
 & \mathbb{C}\,(\mathrm{X} \times (\mathrm{A} + \mathrm{B}) \to \mathrm{Z}) \\
\cong & [\text{product symmetry}] \\
 & \mathbb{C}\,((\mathrm{A} + \mathrm{B}) \times \mathrm{X} \to \mathrm{Z}) \\
\cong & [\text{currying}] \\
 & \mathbb{C}\,(\mathrm{A} + \mathrm{B} \to \mathrm{X} \supset \mathrm{Z}) \\
\cong & [\text{uncotupling}] \\
 & \mathbb{C}\,(\mathrm{A} \to \mathrm{X} \supset \mathrm{Z}) \times \mathbb{C}\,(\mathrm{B} \to \mathrm{X} \supset \mathrm{Z}) \\
\cong & [\text{uncurrying}] \\
 & \mathbb{C}\,(\mathrm{A} \times \mathrm{X} \to \mathrm{Z}) \times \mathbb{C}\,(\mathrm{B} \times \mathrm{X} \to \mathrm{Z}) \\
\cong & [\text{product symmetry}] \\
 & \mathbb{C}\,(\mathrm{X} \times \mathrm{A} \to \mathrm{Z}) \times \mathbb{C}\,(\mathrm{X} \times \mathrm{B} \to \mathrm{Z}) \\
\cong & [\text{cotupling}] \\
 & \mathbb{C}\,((\mathrm{X} \times \mathrm{A}) + (\mathrm{X} \times \mathrm{B}) \to \mathrm{Z})
\end{array}
$$

3.5.2 On Classical Proof Theory

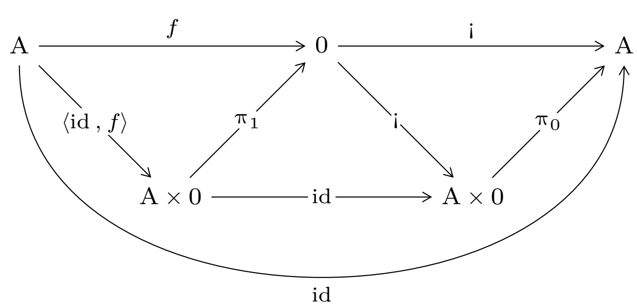
Bicartesian closed categories are ideally suited to interpreting *intuitionistic* propositional logic. However, if we try to use them to interpret *classical* propositional logic, things begin to collapse.

Lemma 3.5.2.1 In a bicartesian closed category, if there is an arrow $f : \mathrm{A} \longrightarrow 0$ then $\mathrm{A} \cong 0$.

Proof. Because $\mathrm{A} \times 0 \cong 0$, the object $\mathrm{A} \times 0$ is initial and the hom set $\mathrm{A} \times 0 \longrightarrow \mathrm{A} \times 0$ is a singleton. In particular, the following diagram must commute:



By pasting, we get $f \cdot ¡(\mathrm{A}) = \mathrm{id}(\mathrm{A})$:



And by the universal property of 0, we also have $¡(\mathrm{A}) \cdot f = \mathrm{id}(0)$. So $\mathrm{A} \cong 0$. $\square$

Corollary 3.5.2.2 In a bicartesian closed category, there is at most one arrow in the hom set $\mathrm{X} \times \mathrm{A} \longrightarrow 0$, and hence in $\mathrm{X} \longrightarrow \mathrm{A} \supset 0$.

This means that if we interpret the proofs of propositional logic as arrows of a bicartesian closed category, then there is at most one proof of a negated proposition.

If we were to adopt the boolean axiom $\neg\neg\mathrm{A} \;\vdash\; \mathrm{A}$, then every proposition would be equivalent to a negated one so there would be at most one proof of any proposition!

In other words, a "boolean category", that is, a bicartesian closed category where $\mathrm{A} \cong (\mathrm{A} \supset 0) \supset 0$ must be a *preorder category*. Such a category is known as a **boolean algebra**.

Chapter 4

Two Dimensional Structure

We have deliberately presented the universal constructions of a bicartesian closed category in such a way as to highlight the parallels between them. The reader may be wondering whether there is some more general construction lurking in the wings, in terms of which terminal and initial objects, products, coproducts and exponentials can all be described. This is indeed the case: the mystery construction is called an *adjunction*. But in order to describe it we will need to first understand some 2-dimensional category theory.

4.1 Naturality

Naturality is the carrier of the two-dimensional structure of categories of categories. It is a sort of *coherence* or *uniformity* property that allows us to interpret a family of features within a particular category as a single feature in the ambient category of categories.

4.1.1 Natural Transformations

Definition 4.1.1.1 (natural transformation) For parallel functors, $\mathrm{F}, \mathrm{G} : \mathbb{C} \longrightarrow \mathbb{D}$, a **natural transformation**, φ from F to G is a functor $\varphi : \mathbb{C} \longrightarrow \mathbb{D}^{\rightarrow}$ such that

$$\varphi \cdot dom = \mathrm{F} \qquad \text{and} \qquad \varphi \cdot cod = \mathrm{G}$$

Explicitly, this means that,

- for each object $\mathrm{A} : \mathbb{C}$ there is an arrow,

  $$\varphi(\mathrm{A}) : \mathbb{D}\,(\mathrm{F}(\mathrm{A}) \to \mathrm{G}(\mathrm{A}))$$

  called the **component** of φ at A, and

- for each arrow $f : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$ there is a **naturality square** witnessing,

$$\mathrm{F}(f) \cdot \varphi(\mathrm{B}) \;=\; \varphi(\mathrm{A}) \cdot \mathrm{G}(f)$$

One way to think about this is that a functor "projects" an image of its domain category into its codomain category. In this sense, a functor acts as a lens, which may "distort" the structure of the source category by identifying distinct objects or arrows. Under this interpretation, a component of a natural transformation acts as a "homotopy" between the images of an object cast by two parallel functors, and a naturality square ensures that these object homotopies are consistent with the images of arrows.

You may also think of the naturality square above as a "2-dimensional arrow" from the 1-dimensional arrow $\mathrm{F}(f)$ to the 1-dimensional arrow $\mathrm{G}(f)$, acting as the "component" of φ at f. But in an ordinary (1-dimensional) category, the only "2-dimensional arrows" available are the trivial ones – i.e. equalities.

4.1.2 Functor Categories

Natural transformations provide a notion of arrows for a collection of parallel functors, turning a hom *set* into a hom *category*:

Definition 4.1.2.1 (functor category) For categories $\mathbb{C}$ and $\mathbb{D}$, define the **functor category**, "$\textsc{Fun}(\mathbb{C}\,,\mathbb{D})$", to have the following structure:

objects $\textsc{Fun}(\mathbb{C}\,,\mathbb{D})_0 \;:=\;$ functors from $\mathbb{C}$ to $\mathbb{D}$

arrows $\textsc{Fun}(\mathbb{C}\,,\mathbb{D})\,(\mathrm{F} \to \mathrm{G}) \;:=\;$ natural transformations from F to G

identities $\mathrm{id}(\mathrm{F})(\mathrm{A}) \;:=\; \mathrm{id}(\mathrm{F}(\mathrm{A}))$
(A component of an *identity natural transformation* is an identity arrow.)

composition $(\varphi \cdot \psi)(\mathrm{A}) \;:=\; \varphi(\mathrm{A}) \cdot \psi(\mathrm{A})$
(A component of a *composite natural transformation* is the composition of the constituent components.)

Composition of natural transformations in the functor category $\textsc{Fun}(\mathbb{C}\,,\mathbb{D})$ is associative and unital just because composition of morphisms is so in $\mathbb{D}$.

An important class of natural transformations is the natural isomorphisms. A **natural isomorphism** is simply an *isomorphism* in a functor category. Unpacking this a bit, for parallel functors, $F , G : \mathbb{C} \longrightarrow \mathbb{D}$, a natural transformation $\varphi : \textsc{Fun}(\mathbb{C} , \mathbb{D})\,(F \to G)$ is a natural isomorphism if it has an *inverse* $\varphi^{-1} : \textsc{Fun}(\mathbb{C} , \mathbb{D})\,(G \to F)$.

Exercise 4.1.2.2 Show that a natural transformation is a (natural) isomorphism in the functor category $\textsc{Fun}(\mathbb{C} , \mathbb{D})$ just in case each of its components is an isomorphism in $\mathbb{D}$.

An *exponential* object $A \supset B$ in a category $\mathbb{C}$ is an object representing the collection of morphisms $\mathbb{C}\,(A \to B)$. If $\mathbb{C}$ is a category of categories, then an object in $\mathbb{C}$ is a category and the morphisms between any two such are functors. So it is natural to wonder whether functor categories are exponential objects in categories of categories. This is generally the case when such exponential categories exist. In particular, it is true in the category of small categories:

Fact 4.1.2.3 (exponential categories) The category $\textsc{Cat}$ has functor categories as exponential objects. That is, for $\mathbb{A} , \mathbb{B} : \textsc{Cat}$,

$$\mathbb{A} \supset \mathbb{B} \quad = \quad \textsc{Fun}(\mathbb{A} , \mathbb{B})$$

From now on we will use exponential notation for functor categories.

4.2 2-Categories

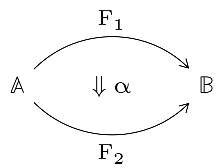
4.2.1 2-Dimensional Categorical Structure

Something subtle and profound has just happened, so let's go through it carefully. Recall that when we introduced categories, we gave them structure at two different dimensions:

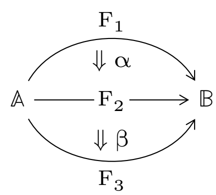
- at dimension 0, we have "points", in the form of objects,
- and at dimension 1, we have "lines", in the form of arrows.

But in introducing natural transformations, we just said that for any two fixed objects, we have a whole *category* of functors and natural transformations between them, so the hom collections in $\textsc{Cat}$ are not 0-dimensional sets, but rather 1-dimensional categories. This gives the category $\textsc{Cat}$ structure at dimension 2 as well!

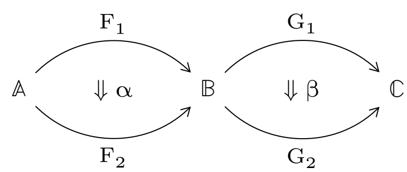
For fixed categories $\mathbb{A}$ and $\mathbb{B}$, and parallel functors $F_1 , F_2 : \mathbb{A} \longrightarrow \mathbb{B}$, we can draw a natural transformation $\alpha : F_1 \longrightarrow F_2$ as a "surface" in a 2-dimensional diagram in Cat:



And if we have another parallel functor $F_3 : \mathbb{A} \longrightarrow \mathbb{B}$ and natural transformation $\beta : F_2 \longrightarrow F_3$, we can draw the composite natural transformation $\alpha \cdot \beta : F_1 \longrightarrow F_3$ as:



Now we come to the question of what happens if we don't require the pair of objects under consideration to remain fixed. Consider the following pair of natural transformations:



We have the composite parallel functors,

$$(F_1 \cdot G_1), (F_2 \cdot G_2) : \mathbb{A} \longrightarrow \mathbb{C}$$

Is there some way to form a composite natural transformation, $\alpha \cdot\cdot \beta : F_1 \cdot G_1 \longrightarrow F_2 \cdot G_2$? [1] Well, given an object $A : \mathbb{A}$, we know that the component of such a composite natural transformation at A should be an arrow,

$$(\alpha \cdot \cdot \beta)(A) \ : \ \mathbb{C}\left((G_1 \circ F_1)(A) \to (G_2 \circ F_2)(A)\right)$$

Here is how we can define it:

Lemma 4.2.1.1 (horizontal composition of natural transformations) In the situation just described, there is a natural transformation

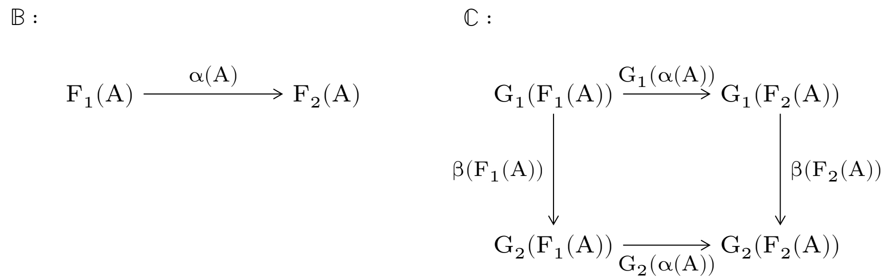
$$\alpha \cdot \cdot \beta \ : \ F_1 \cdot G_1 \longrightarrow F_2 \cdot G_2$$

called the **horizontal composition** of α and β, with components,

$$(\alpha \cdot \cdot \beta)(A) \ := \ G_1(\alpha(A)) \cdot \beta(F_2(A)) \ = \ \beta(F_1(A)) \cdot G_2(\alpha(A))$$

[1] Read "$\alpha \cdot \cdot \beta$" as "α beside β" – that is, composed along their common boundary two dimensions down, rather than the usual one dimension down with "$- \cdot -$". The reader may amuse herself thinking about how to extend this pattern to still higher dimensions.

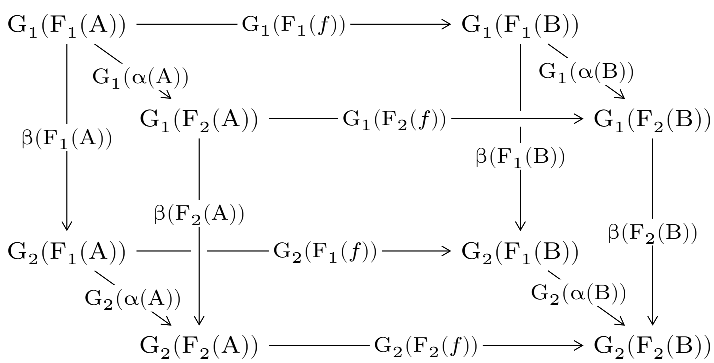
Proof. To see that the two composites are indeed equal consider the component of α at A in $\mathbb{B}$. This determines a naturality square for β at $\alpha(A)$ in $\mathbb{C}$:



(4.1)

establishing that the two expressions for the putative components of $\alpha \cdot\cdot \beta$ coincide.

Now we must show that this definition of components respects arrows in $\mathbb{A}$. Let $f : \mathbb{A}(A \to B)$. Then in the diagram in $\mathbb{C}$,



- the left and right squares are the naturality squares of β at $\alpha(A)$ and $\alpha(B)$,
- the top and bottom squares are G_1 and G_2 functor-images of the naturality squares of α at f,
- and the back and front squares are the naturality squares of β at $F_1(f)$ and $F_2(f)$.

Pasting the top and front – or equivalently, of the back and bottom – squares establishes the naturality of $\alpha \cdot\cdot \beta$ at f. □

Remark 4.2.1.2 (connection to arrow categories) The construction in the preceding proof should remind you of the double arrow category construction. This is because the *arrow category* $\mathbb{D}^{\to}$ is equivalent to the functor category $\mathbb{I} \supset \mathbb{D}$, where $\mathbb{I}$ is the *interval category*.

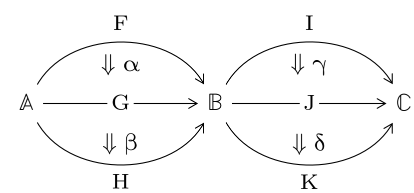
Exercise 4.2.1.3 Check that horizontal composition respects the composition structure of each functor category by pasting squares onto diagram 4.1.

Mercifully, we do not need to reason about natural transformations in this cumbersome, component-wise manner. After all, the whole point of the categorical approach is to allow us to reason behaviorally rather than structurally. The component-wise definition of natural transformations is reminiscent of Plato's *Allegory of the Cave*: the components and naturality squares in the codomain category are mere shadows cast by the flesh-and-blood natural transformations, which are two-dimensional morphisms between parallel one-dimensional morphisms living in a 2-dimensional category. Let us turn our heads and stumble into the light.

A strict 2-dimensional globular category, or **2-category**, has:

- 0-dimensional structure in the form of objects or **0-cells**,
- 1-dimensional structure in the form of arrows or **1-cells** between 0-cells. These compose along their 0-dimensional boundaries such that any path of 1-cells has a unique composite.
- 2-dimensional structure in the form of surfaces or **2-cells** between parallel 1-cells. These compose along both their 1- and 0-dimensional boundaries such that any 2-dimensional pasting diagram has a unique composite.

The category Cat with its categories, functors and natural transformations, is a 2-category. In particular, this implies the **interchange law**, which says that in the situation,



we have:

$$(\alpha \cdot \beta) \cdot\cdot (\gamma \cdot \delta) \;=\; (\alpha \cdot\cdot \gamma) \cdot (\beta \cdot\cdot \delta) \;:\; F \cdot I \longrightarrow H \cdot K$$

There are several equivalent ways to axiomatize the notion of 2-category, but for our purposes, this informal description will suffice.

4.2.2 String Diagrams

If we take the Poincaré (or graph) dual of the 2-dimensional diagrams we have been drawing, we obtain a very useful graphical language for 2-categories, called **string diagrams**. Specifically, we will use "planar progressive" string diagrams to represent configurations of 0-, 1-, and 2-cells.

In the graphical language of string diagrams,

0-cells (e.g. $\mathbb{C}$: Cat) are represented as regions in the plane,

1-cells (e.g. $F : \mathbb{C} \longrightarrow \mathbb{D}$) are represented by lines or "strings" or "wires", (in our convention) progressing from top to bottom,

2-cells (e.g. $\alpha : F \longrightarrow G$) are represented by points, fattened up into "nodes" or "beads" with the wires representing their domain 1-cells entering from above, and those representing their codomain 1-cells exiting from below.

identity 1-cells (e.g. $\mathrm{id}(\mathbb{C})$) are represented by a "ghost wire", or usually, not at all,

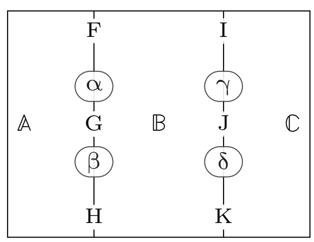
1-cell composition (e.g. $F \cdot I$) is represented by juxtaposing wires side-by-side, separated by the region representing their common boundary,

identity 2-cells (e.g. $\mathrm{id}(F)$) are represented by a "ghost node", or usually, not at all,

2-cell vertical composition (e.g. $\alpha \cdot \beta$) is represented by wiring the output sockets of the first node to the input sockets of the second,

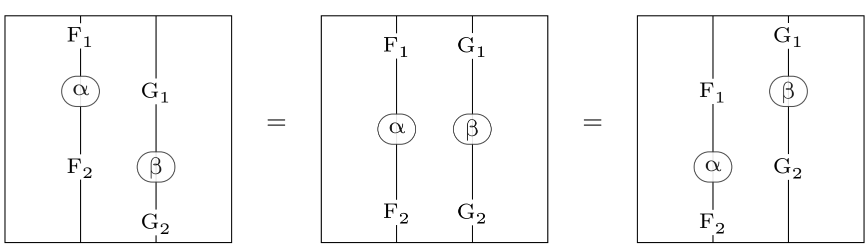
2-cell horizontal composition (e.g. $\alpha \cdot\cdot \gamma$) is represented juxtaposing nodes , together with their wires, side-by-side,

As an example, the pasting diagram for the interchange law becomes the string diagram:



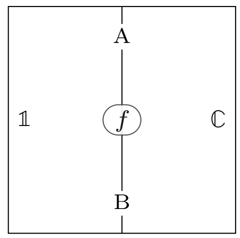
Typically, we omit labeling the regions as their identities can always be inferred.

The commuting of diagram 4.1 in the definition of horizontal composition of 2-cells represents the notion of **naturality as independence**, depicted in string diagrams by the property that two beads without an output-to-input connection between them may freely "slide" past one another along their wires, and it makes no difference which is above or below the other.



The idea of independence is the heart of naturality; the rest of its properties can be recovered from this.

There is a handy "trick" for transforming diagrams within a category into string diagrams using *global elements*. Notice that for any object $\mathrm{A} : \mathbb{C}$ there is a functor (which we typically overload with the same name) $\mathrm{A} : \mathbb{1} \longrightarrow \mathbb{C}$ picking out that object, and for any arrow $f : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$ there is a natural transformation between the respective functors, which we can represent as the string diagram:



Exercise 4.2.2.1 Show how the commuting of *naturality squares* is a consequence of naturality as independence.

4.3 Adjunctions

Adjunctions are constructions that may exist in the context of a 2-dimensional category. In any 2-category adjunctions have a behavioral or "external" characterization. In the 2-category Cat they also have structural or "internal" characterizations. While the behavioral characterization is more perspicuous, the structural ones will be important for the semantics of proof theory.

4.3.1 Behavioral Characterization

For our primary definition of adjunction we adopt the following behavioral one, which we call the "zigzag characterization":

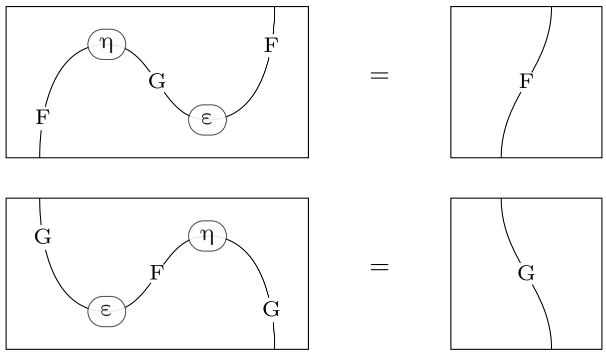
Definition 4.3.1.1 (adjunction – zigzag characterization) Anti-parallel functors $\mathrm{F} : \mathbb{C} \longrightarrow \mathbb{D}$ and $\mathrm{G} : \mathbb{D} \longrightarrow \mathbb{C}$ form an **adjunction**, written "$\mathrm{F} \dashv \mathrm{G}$", if there exist natural transformations:

- $\eta \;:\; \mathrm{id}(\mathbb{C}) \longrightarrow \mathrm{F} \cdot \mathrm{G}$, called the adjunction's **unit**, and
- $\varepsilon \;:\; \mathrm{G} \cdot \mathrm{F} \longrightarrow \mathrm{id}(\mathbb{D})$, called the adjunction's **counit**,

satisfying the relations:

- **left zigzag law**: $(\eta \cdot \cdot \mathrm{F}) \cdot (\mathrm{F} \cdot \cdot \varepsilon) \;=\; \mathrm{id}(\mathrm{F})$
- **right zigzag law**: $(\mathrm{G} \cdot \cdot \eta) \cdot (\varepsilon \cdot \cdot \mathrm{G}) \;=\; \mathrm{id}(\mathrm{G})$

The reason for the name “zigzag” becomes apparent when the laws are drawn as string diagrams:



The chirality of the zigzag laws comes from the fact that when $F \dashv G$, F is called **left adjoint** to G, and G is called **right adjoint** to F.

4.3.2 Structural Characterizations

Instead of thinking of an adjunction as a single structure living in the 2-category Cat, we can think of it as a correlation between families of structures living in two particular categories. This is like the component-wise presentation of a *natural transformation*. One such characterization of an adjunction is the following:

Definition 4.3.2.1 (adjunction – natural bijection of hom sets charaterization) Anti-parallel functors $F : \mathbb{C} \longrightarrow \mathbb{D}$ and $G : \mathbb{D} \longrightarrow \mathbb{C}$ form an adjunction $F \dashv G$ if for any $A : \mathbb{C}$ and $B : \mathbb{D}$ there is a natural bijection of hom sets:

$$\frac{\mathbb{C}\,(A \to G(B))}{\mathbb{D}\,(F(A) \to B)}\ \theta$$

This characterization is internal or structural because, unlike the zigzag characterization, we look inside the categories $\mathbb{C}$ and $\mathbb{D}$. We will call the downward direction of such a bijection “$\_^\sharp$” and the upward direction “$\_^\flat$”, so:

$$\begin{array}{ccc} f : \mathbb{C}\,(A \to G(B)) & \overset{\_^\sharp}{\longmapsto} & f^\sharp : \mathbb{D}\,(F(A) \to B) \\ & \text{and} & \\ g : \mathbb{D}\,(F(A) \to B) & \overset{\_^\flat}{\longmapsto} & g^\flat : \mathbb{C}\,(A \to G(B)) \\ & \text{and} & \\ (f^\sharp)^\flat = f & \text{and} & (g^\flat)^\sharp = g \end{array}$$

We call the image of an arrow under this bijection its **adjoint complement**.

A bijection of hom sets is *natural* if it extends along its boundary by the relevant functors. In this case, that means that for any $a : \mathbb{C}\,(\mathrm{A}' \to \mathrm{A})$ and $b : \mathbb{D}\,(\mathrm{B} \to \mathrm{B}')$ we have,

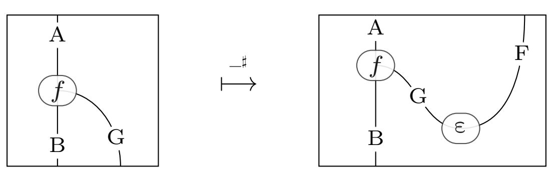
$$\begin{array}{ll} \mathbb{C} : & \mathrm{A}' \xrightarrow{\;a\;} \mathrm{A} \xrightarrow{\;f = g^\flat\;} \mathrm{G}(\mathrm{B}) \xrightarrow{\;\mathrm{G}(b)\;} \mathrm{G}(\mathrm{B}') \\ \hline\hline \mathbb{D} : & \mathrm{F}(\mathrm{A}') \xrightarrow[\mathrm{F}(a)]{} \mathrm{F}(\mathrm{A}) \xrightarrow[g = f^\sharp]{} \mathrm{B} \xrightarrow[b]{} \mathrm{B}' \end{array}$$

Technically, this is a *natural isomorphism* in the functor category $(\mathbb{C}^\circ \times \mathbb{D}) \supset \mathrm{S{\scriptsize ET}}$ between $\mathbb{C}\,(\underline{1} \to \mathrm{G}(\underline{2}))$ and $\mathbb{D}\,(\mathrm{F}(\underline{1}) \to \underline{2})$.

We won't prove the equivalence of the various characterizations of adjunctions in this course, but the fact that the zigzag characterization implies the natural bijection of hom sets characterization is easy to see using string diagrams.

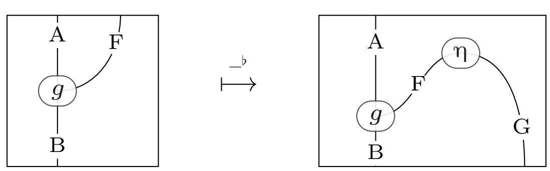
Given an arrow $f : \mathbb{C}\,(\mathrm{A} \to \mathrm{G}(\mathrm{B}))$, the obvious way to construct an arrow $f^\sharp : \mathbb{D}\,(\mathrm{F}(\mathrm{A}) \to \mathrm{B})$ out of the parts at hand is by defining,

$$f^\sharp \quad := \quad \mathrm{F}(f) \cdot \varepsilon(\mathrm{B})$$

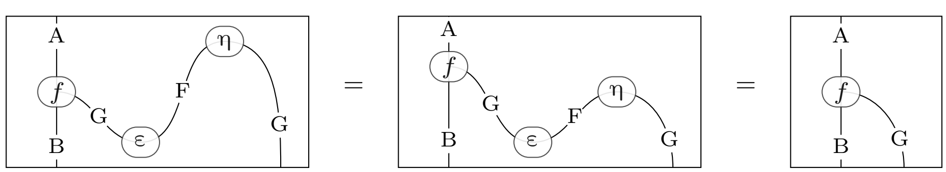


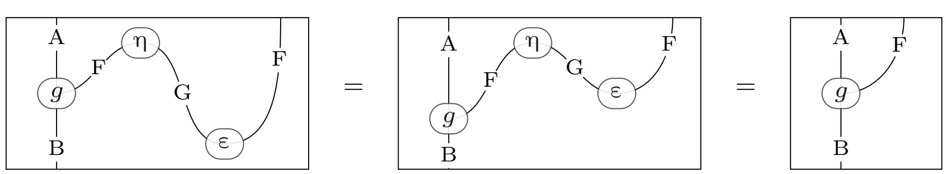
Likewise, given an arrow $g : \mathbb{D}\,(\mathrm{F}(\mathrm{A}) \to \mathrm{B})$, the obvious way to construct an arrow $g^\flat : \mathbb{C}\,(\mathrm{A} \to \mathrm{G}(\mathrm{B}))$ is by defining,

$$g^\flat \quad := \quad \eta(\mathrm{A}) \cdot \mathrm{G}(g)$$



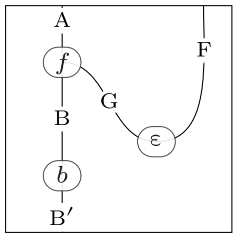
We can use the zigzag laws to show that $-^\sharp$ and $-^\flat$ are inverse operations:

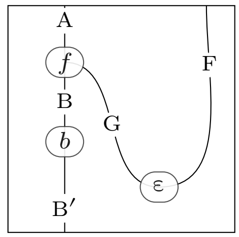
$(f^\sharp)^\flat \quad =$ 

$(g^\flat)^\sharp \quad =$ 

The naturality of the bijection in the domain and codomain coordinates is also obvious in the graphical language.

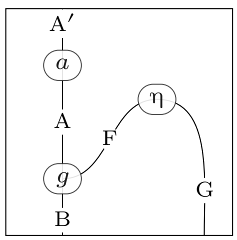
- For naturality in the codomain coordinate, given $b : \mathbb{D}\,(\mathrm{B} \to \mathrm{B}')$ we have:

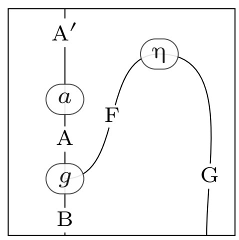
$f^\sharp{\cdot}b \quad =$ 

$=$ 

$= \quad (f{\cdot}\mathrm{G}(b))^\sharp$

- For naturality in the domain coordinate, given $a : \mathbb{C}\,(\mathrm{A}' \to a)$ we have:

$a{\cdot}g^\flat \quad =$ 

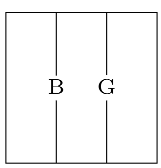
$=$ 

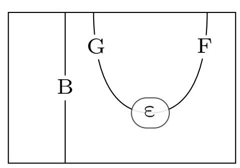
$= \quad (\mathrm{F}(a){\cdot}g)^\flat$

Exercise 4.3.2.2 With f, g, a and b as in the proof above, verify the following:

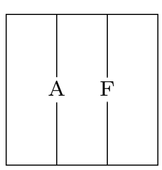
- $(g \cdot b)^\flat \quad = \quad g^\flat \cdot \mathrm{G}(b)$
- $(a \cdot f)^\sharp \quad = \quad \mathrm{F}(a) \cdot f^\sharp$

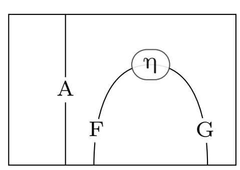
We can work out that the adjoint complements of components of the unit and counit of an adjunction are identities, which are easily seen in string diagrams:

$\mathrm{id}(\mathrm{G}(\mathrm{B})) \quad =$ 

$\overset{\_^\sharp}{\longmapsto}$ 

$= \quad \varepsilon(\mathrm{B})$

$\mathrm{id}(\mathrm{F}(\mathrm{A})) \quad =$ 

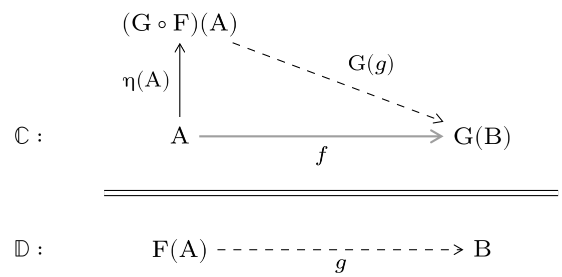
$\overset{\_^\flat}{\longmapsto}$ 

$= \quad \eta(\mathrm{A})$

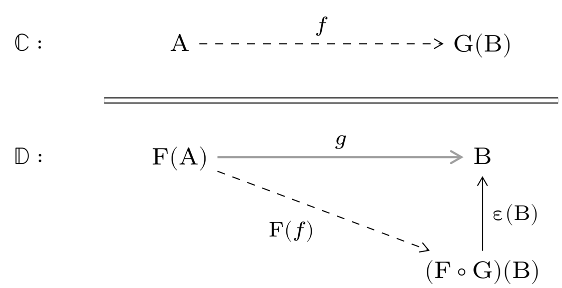
Two more structural characterizations of an adjunction – which are dual to one anther – are given by the universal properties of its unit and counit.

Definition 4.3.2.3 (adjunction – universal property of unit characterization) Anti-parallel functors $\mathrm{F} : \mathbb{C} \longrightarrow \mathbb{D}$ and $\mathrm{G} : \mathbb{D} \longrightarrow \mathbb{C}$ form and adjunction

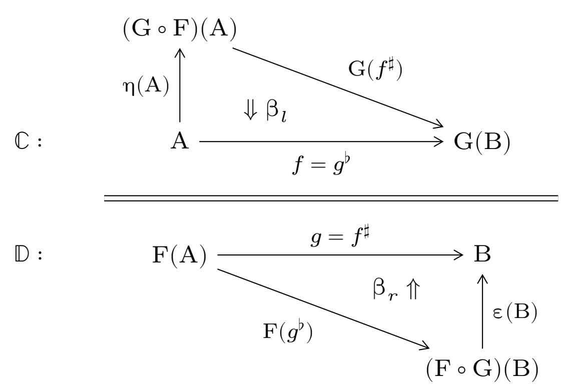
F ⊣ G if there is a natural transformation $\eta : \mathrm{id}(\mathbb{C}) \longrightarrow F \cdot G$ such that for any $f : \mathbb{C}\,(A \to G(B))$ there is a unique $g : \mathbb{D}\,(F(A) \to B)$ such that $\eta(A) \cdot G(g) = f$:



Definition 4.3.2.4 (adjunction – universal property of counit characterization) Anti-parallel functors $F : \mathbb{C} \longrightarrow \mathbb{D}$ and $G : \mathbb{D} \longrightarrow \mathbb{C}$ form and adjunction F ⊣ G if there is a natural transformation $\varepsilon : G \cdot F \longrightarrow \mathrm{id}(\mathbb{D})$ such that for any $g : \mathbb{D}\,(F(A) \to B)$ there is a unique $f : \mathbb{C}\,(A \to G(B))$ such that $F(f) \cdot \varepsilon(B) = g$:



Of course, $g = f^\sharp$ and $f = g^\flat$. So the "internal picture" of an adjunction looks like this:



The 2-cells labeled "β_l" and "β_r" are both equalities because $\mathbb{C}$ and $\mathbb{D}$ are just ordinary (1-dimensional) categories so equality is the only possible kind of 2-cell. But as we will see shortly, it is convenient to give them a suggestive name and orientation.

4.3.3 Harmony of the Propositional Connectives

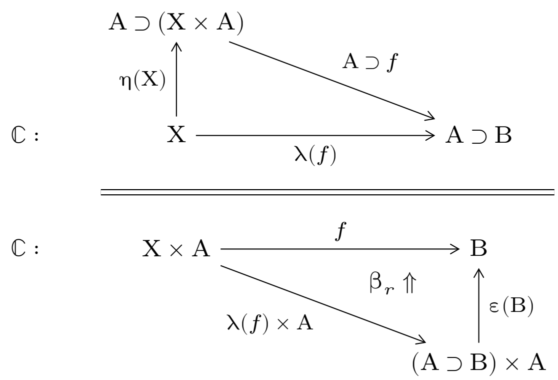
The universal property of the counit should immediately remind you of the definition of an *exponential* object. Indeed, for a fixed object A, we have endofunctors

$$- \times \mathrm{A} \quad : \quad \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{and} \qquad \mathrm{A} \supset - \quad : \quad \mathbb{C} \longrightarrow \mathbb{C}$$

and an adjunction

$$- \times \mathrm{A} \quad \dashv \quad \mathrm{A} \supset -$$

The counit of this adjunction is the *evaluation map* and the unit is the *pairing map*. The internal picture of this adjunction looks like this:



Recall our interpretations for the natural deduction introduction and elimination rules for implication in terms of exponentials:

$$[\![\supset +]\!] \quad = \quad \lambda(-) \qquad \text{and} \qquad [\![\supset -]\!] \quad = \quad \varepsilon(-)$$

The universal property of exponentials is just the universal property of the counit of this adjunction. Its commuting triangle β_r,

$$\lambda(f) \times \mathrm{A} \cdot \varepsilon(\mathrm{B}) \quad = \quad f$$

expresses the local reduction for implication:

$$\dfrac{\dfrac{\Gamma \qquad \dfrac{\dfrac{\overline{\Gamma} \qquad \overline{\mathrm{A}}}{\mathcal{D}}}{\mathrm{B}}}{\mathrm{A} \supset \mathrm{B}}\supset+ \qquad \mathrm{A}}{\mathrm{B}}\supset- \qquad \overset{\supset>}{\longmapsto} \qquad \dfrac{\dfrac{\Gamma \qquad \mathrm{A}}{\mathcal{D}}}{\mathrm{B}}$$

Identity expansion for exponentials (lemma 3.4.1.2), which gives us the adjoint complement of an identity morphism,

$$\mathrm{id}(\mathrm{A} \supset \mathrm{B}) \quad = \quad \lambda(\varepsilon)$$

expresses the local expansion of an identity derivation:

$$\mathrm{A} \supset \mathrm{B} \quad \overset{\supset<}{\longmapsto} \quad \dfrac{\mathrm{A} \supset \mathrm{B} \qquad \dfrac{\overline{\mathrm{A} \supset \mathrm{B}} \qquad \overline{\mathrm{A}}}{\mathrm{B}}\supset-}{\mathrm{A} \supset \mathrm{B}}\supset+$$

Pre-composing with a curry (exercise 3.4.1.5), which expresses the naturality of the hom set bijection in its domain coordinate,

$$i \cdot \lambda f \;\; = \;\; \lambda(i \times \mathrm{A} \cdot f)$$

allows us to move any derivation pre-composed with a $\supset+$ rule into the hypothetical subderivation. We can view this as a kind of permutation conversion:

$$\dfrac{\dfrac{\dfrac{\Gamma}{\mathcal{E}}}{\mathrm{C}} \qquad \dfrac{\dfrac{\overline{\mathrm{C}} \quad \overline{\mathrm{A}}}{\mathcal{D}}}{\mathrm{B}}}{\mathrm{A} \supset \mathrm{B}}\supset+ \quad \overset{\supset\rightleftarrows}{\longmapsto} \quad \dfrac{\Gamma \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\Gamma}}{\mathcal{E}}}{\mathrm{C}} \quad \overline{\mathrm{A}}}{\mathcal{D}}}{\mathrm{B}}}{\mathrm{A} \supset \mathrm{B}}\supset+$$

Combining this with the special case of local expansion of an identity derivation yields the general case of local expansion for implication.

Remarkably, the same pattern obtains for the other negatively-presented propositional connectives ($\wedge$ and $\top$). There is an adjunction $\mathrm{F} \dashv \mathrm{G}$ such that:

- their introduction rule is interpreted by the adjoint complement operation,

$$[\![*+]\!] \quad := \quad -^{\flat}$$

- their elimination rule is interpreted by the component of the counit,

$$[\![*-]\!] \quad := \quad \varepsilon(-)$$

- their local reduction is interpreted by the commuting triangle in the universal property of the counit,

$$[\![* >]\!] \quad := \quad \beta_r$$

- their identity local expansion is interpreted by the adjoint complement of an identity morphism,

$$[\![* <]\!] \quad := \quad \mathrm{id}(\mathrm{G}\, -) = (\varepsilon\, -)^{\flat}$$

- their permutation conversion is interpreted by the naturality of the hom set bijection in the domain coordinate,

$$[\![* \rightleftarrows]\!] \quad := \quad i \cdot g^\flat = (\mathrm{F}(i) \cdot g)^\flat$$

And just as remarkably, modulo *context distributivity*, the dual pattern obtains for the positively-presented propositional connectives ($\vee$ and $\bot$). There is an adjunction such that:

- their introduction rule is interpreted by the component of the unit,

$$[\![*+]\!] \quad := \quad \eta(-)$$

- their elimination rule is interpreted by the adjoint complement operation,

$$[\![*-]\!] \quad := \quad -^\sharp$$

- their local reduction is interpreted by the commuting triangle in the universal property of the unit,

$$[\![* >]\!] \quad := \quad \beta_l$$

- their identity local expansion is interpreted by the adjoint complement of an identity morphism,

$$[\![* <]\!] \quad := \quad \mathrm{id}(\mathrm{F}\ -) = (\eta\ -)^\sharp$$

- their permutation conversion is interpreted by the naturality of the hom set bijection in the codomain coordinate,

$$[\![* \rightleftarrows]\!] \quad := \quad f^\sharp \cdot i = (f \cdot \mathrm{G}(i))^\sharp$$

This provides the *algebraic basis for connective harmony.*

Adjoints to a Diagonal Functor

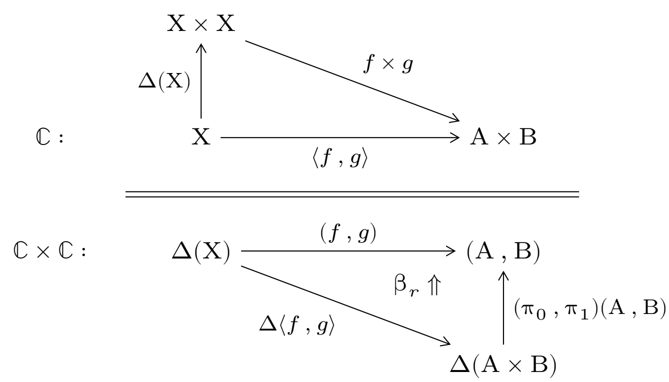
Recall that the category Cat has *products* via the *product category* construction, and that in any category with products we can define *diagonal maps*. This implies that for any category $\mathbb{C}$ we have a **diagonal functor**,

$$\begin{array}{ccc} & \underline{\Delta} & \\ \mathbb{C} & \longrightarrow & \mathbb{C} \times \mathbb{C} \\ \mathrm{A} & \longmapsto & (\mathrm{A}\,,\mathrm{A}) \\ f & \longmapsto & (f\,,f) \end{array}$$

When the category $\mathbb{C}$ has products, the *product functor*, $- \times - : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$, is right adjoint to the diagonal functor:

$$\Delta \quad \dashv \quad - \times -$$

The counit of this adjunction is the ordered pair of *projections* and the unit is the *diagonal map*. The internal picture for this adjunction looks like this:



Recall our interpretations for the natural deduction introduction and elimination rules for conjunction in terms of products:

$$[\![\wedge+]\!] \quad = \quad \langle - , - \rangle \qquad \text{and} \qquad [\![\wedge-]\!] \quad = \quad (\pi_0 , \pi_1)$$

The universal property of products is just the universal property of the counit of this adjunction. Its commuting triangle β_r,

$$\Delta\langle f , g \rangle \cdot (\pi_0 , \pi_1) \quad = \quad (f , g)$$

expresses the local reduction for conjunction:

$$\dfrac{\dfrac{\Gamma \quad \dfrac{\dfrac{\overline{\Gamma}}{\mathcal{D}_0}}{A} \quad \dfrac{\dfrac{\overline{\Gamma}}{\mathcal{D}_1}}{B}}{A \wedge B}\,\wedge+}{A}\,\wedge-_0 \qquad \dfrac{\dfrac{\Gamma \quad \dfrac{\dfrac{\overline{\Gamma}}{\mathcal{D}_0}}{A} \quad \dfrac{\dfrac{\overline{\Gamma}}{\mathcal{D}_1}}{B}}{A \wedge B}\,\wedge+}{B}\,\wedge-_1 \qquad \overset{\wedge>}{\longmapsto} \qquad \dfrac{\dfrac{\Gamma}{\mathcal{D}_0}}{A} \quad \dfrac{\dfrac{\Gamma}{\mathcal{D}_1}}{B}$$

Identity expansion for products (lemma 3.2.1.2), which gives us the adjoint complement of an identity morphism,

$$\mathrm{id}(A \times B) \quad = \quad \langle \pi_0 , \pi_1 \rangle$$

expresses the local expansion of an identity derivation:

$$A \wedge B \qquad \overset{\wedge<}{\longmapsto} \qquad \dfrac{A \wedge B \quad \dfrac{\overline{A \wedge B}}{A}\,\wedge-_0 \quad \dfrac{\overline{A \wedge B}}{B}\,\wedge-_1}{A \wedge B}\,\wedge+$$

Pre-composing with a tuple (exercise 3.2.1.5), which expresses the naturality of the hom set bijection in its domain coordinate,

$$i \cdot \langle f \,, g \rangle \quad = \quad \langle i \cdot f \,, i \cdot g \rangle$$

allows us to take any derivation pre-composed with a $\wedge{+}$ rule, copy it and move it into the hypothetical subderivations. We can view this as a permutation conversion:

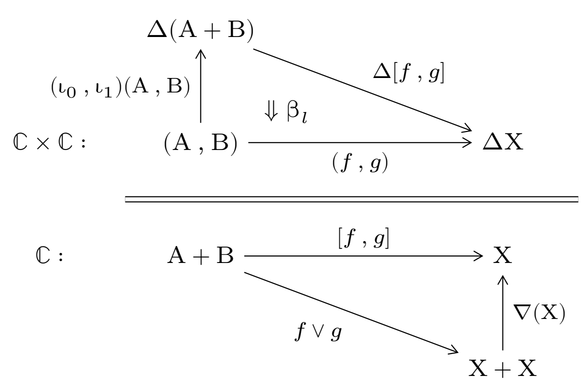
$$\dfrac{\begin{array}{c}\Gamma\\ \hline \mathcal{E}\\ \hline \mathrm{C}\end{array} \quad \begin{array}{c}\overline{\mathrm{C}}\\ \hline \mathcal{D}_1\\ \hline \mathrm{A}\end{array} \quad \begin{array}{c}\overline{\mathrm{C}}\\ \hline \mathcal{D}_2\\ \hline \mathrm{B}\end{array}}{\mathrm{A} \wedge \mathrm{B}}\,\wedge{+} \quad \overset{\wedge\rightleftarrows}{\longmapsto} \quad \dfrac{\Gamma \quad \begin{array}{c}\overline{\Gamma}\\ \hline \mathcal{E}\\ \hline \mathrm{C}\\ \hline \mathcal{D}_1\\ \hline \mathrm{A}\end{array} \quad \begin{array}{c}\overline{\Gamma}\\ \hline \mathcal{E}\\ \hline \mathrm{C}\\ \hline \mathcal{D}_2\\ \hline \mathrm{B}\end{array}}{\mathrm{A} \wedge \mathrm{B}}\,\wedge{+}$$

Combining this with the special case of local expansion of an identity derivation yields the general case of local expansion for conjunction.

When the category $\mathbb{C}$ has coproducts, the *diagonal functor* $\Delta : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{C}$ also has a left adjoint, namely the *coproduct functor* $- + - : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$:

$$- + - \quad \dashv \quad \Delta$$

The unit of this adjunction is the ordered pair of *insertions* and the counit is the *codiagonal map*. The internal picture for this adjunction looks like this:



Recall our interpretations (modulo the *distributive law*) for the natural deduction introduction and elimination rules for disjunction in terms of coproducts:

$$[\![\vee{+}]\!] \quad = \quad (\iota_0 \,, \iota_1) \qquad \text{and} \qquad [\![\vee{-}^{\dagger}]\!] \quad = \quad [- \,, -]$$

The universal property of coproducts is just the universal property of the unit of this adjunction. Its commuting triangle β_l,

$$(\iota_0 \,, \iota_1) \cdot \Delta[f \,, g] \quad = \quad (f \,, g)$$

expresses the local reduction for disjunction:

$$\dfrac{\dfrac{A}{A \vee B}\,\vee+_0 \qquad \dfrac{\overline{A}}{\dfrac{\mathcal{D}_0}{C}} \qquad \dfrac{\overline{B}}{\dfrac{\mathcal{D}_1}{C}}}{C}\,\vee-^{\dagger} \qquad \dfrac{\dfrac{B}{A \vee B}\,\vee+_1 \qquad \dfrac{\overline{A}}{\dfrac{\mathcal{D}_0}{C}} \qquad \dfrac{\overline{B}}{\dfrac{\mathcal{D}_1}{C}}}{C}\,\vee-^{\dagger} \quad \overset{\vee>}{\longmapsto} \quad \dfrac{A}{\dfrac{\mathcal{D}_0}{C}} \qquad \dfrac{B}{\dfrac{\mathcal{D}_1}{C}}$$

Identity expansion for coproducts (lemma 3.3.1.2), which gives us the adjoint complement of an identity morphism,

$$\mathrm{id}(A + B) \quad = \quad [\iota_0 \, , \iota_1]$$

expresses the local expansion of an identity derivation:

$$A \vee B \quad \overset{\vee<}{\longmapsto} \quad \dfrac{A \vee B \qquad \dfrac{\overline{A}}{A \vee B}\,\vee+_0 \qquad \dfrac{\overline{B}}{A \vee B}\,\vee+_1}{A \vee B}\,\vee-^{\dagger}$$

Post-composing with a cotuple (lemma 3.3.1.5), which expresses the naturality of the hom set bijection in its codomain coordinate,

$$[f \, , g] \cdot i \quad = \quad [f \cdot i \, , g \cdot i]$$

allows us to take any derivation post-composed with a $\vee-$ rule, copy it and move it into the hypothetical subderivations. This is the permutation conversion for disjunction:

$$\dfrac{\dfrac{A \vee B \qquad \dfrac{\overline{A}}{\dfrac{\mathcal{D}_0}{C}} \qquad \dfrac{\overline{B}}{\dfrac{\mathcal{D}_1}{C}}}{C}\,\vee-^{\dagger}}{\dfrac{\mathcal{E}}{D}} \quad \overset{\vee\rightleftarrows}{\longmapsto} \quad \dfrac{A \vee B \qquad \dfrac{\overline{A}}{\dfrac{\mathcal{D}_0}{\dfrac{C}{\dfrac{\mathcal{E}}{D}}}} \qquad \dfrac{\overline{B}}{\dfrac{\mathcal{D}_1}{\dfrac{C}{\dfrac{\mathcal{E}}{D}}}}}{D}\,\vee-^{\dagger}$$

Adjoints to a Bang Functor

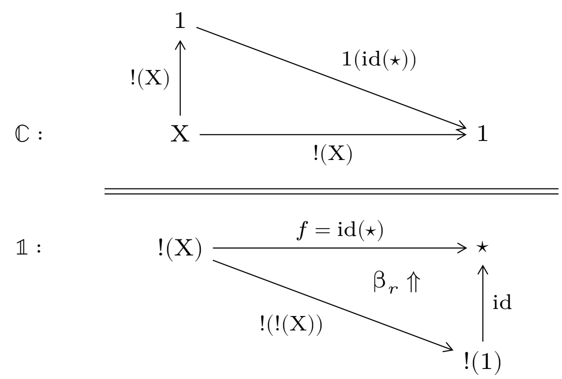
Recall that the category Cat has *terminal objects* in the form of *singleton categories*, and that in any category with a terminal object we have *bang maps*. This implies that for any category $\mathbb{C}$ we have a **bang functor**,

$$\begin{array}{ccc} & \underline{!} & \\ \mathbb{C} & \longrightarrow & \mathbb{1} \\ A & \longmapsto & \star \\ f & \longmapsto & \mathrm{id}(\star) \end{array}$$

When the category $\mathbb{C}$ has terminal objects, the functor $1 : \mathbb{1} \longrightarrow \mathbb{C}$ picking out a terminal object is right adjoint to the bang functor:

$$! \quad \dashv \quad 1$$

The counit of this adjunction is (trivially) the identity and the unit is the *bang map*. The internal picture for this adjunction looks like this:



The triangle in $\mathbb{1}$ is trivial because $\mathrm{id}(\star)$ is the only arrow in the whole category.

Recall our interpretation for the natural deduction introduction rule for truth in terms of terminal objects:

$$[\![\top+]\!] \quad = \quad !(-)$$

This fits the pattern we have been describing if we rewrite it as:

$$\dfrac{\Gamma \qquad \begin{array}{c}\overline{\star}\\ \mathcal{D}\\ \overline{\star}\end{array}}{\top}\ \top+$$

But there is only one possible choice for $[\![\mathcal{D}]\!]$, namely, $\mathrm{id}(\star)$.

The derived "elimination rule for truth" informs us of the existence of the morphism $\mathrm{id}(\star)$ in the category $\mathbb{1}$, which is hardly earth-shattering news. Likewise, the derived "local reduction for truth" tells us that $\mathrm{id}(\star) \cdot \mathrm{id}(\star) = \mathrm{id}(\star)$.

Identity expansion for terminal objects (lemma 3.1.1.2), which gives us the adjoint complement of an identity morphism,

$$\mathrm{id}(1) \quad = \quad !(1)$$

expresses the local expansion of an identity derivation:

$$\top \qquad \overset{\top<}{\longmapsto} \qquad \dfrac{\top \qquad \overline{\star}}{\top}\ \top+$$

Pre-composing with a bang (exercise 3.1.1.4), which is naturality of the hom set bijection in its domain coordinate,

$$i \cdot ! \quad = \quad !$$

allows us to take any derivation pre-composed with a $\top+$ rule, and "throw it away" by moving its !-image – i.e. $\mathrm{id}(\star)$ – to the singleton category. We can view this as a permutation conversion:

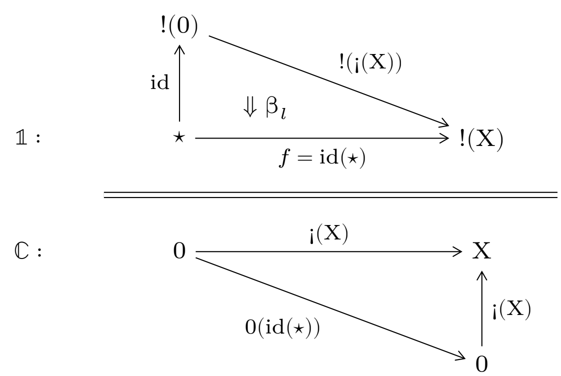
$$\dfrac{\dfrac{\dfrac{\Gamma}{\mathcal{E}}}{\mathrm{A}} \qquad \dfrac{}{\star}}{\top}\ \top+ \qquad \overset{\top\rightleftarrows}{\longmapsto} \qquad \dfrac{\Gamma \qquad \dfrac{}{\star}}{\top}\ \top+$$

Combining this with the special case of local expansion of an identity derivation yields the general case of local expansion for truth.

When the category $\mathbb{C}$ has initial objects, the functor $0 : \mathbb{1} \longrightarrow \mathbb{C}$ picking out an initial object is left adjoint to the bang functor:

$$0 \quad \dashv \quad !$$

The unit of this adjunction is (trivially) the identity and the unit is the *cobang map*. The internal picture for this adjunction looks like this:



Again, the triangle in the category $\mathbb{1}$ is trivial.

Recall our interpretation (modulo the *absorption law*) for the natural deduction elimination rule for falsehood in terms of initial objects:

$$[\![\bot-^{\dagger}]\!] \quad = \quad ¡(-)$$

This fits the pattern we have been describing if we rewrite it as:

$$\dfrac{\bot \qquad \dfrac{\dfrac{\dfrac{}{\star}}{\mathcal{D}}}{\star}}{\mathrm{A}}\ \bot-^{\dagger}$$

where again, the only possibility for $[\![\mathcal{D}]\!]$ is $\mathrm{id}(\star)$. The derived "introduction rule" and "local reduction" for falsehood are as uninformative as the elimination and local reduction for truth.

Identity expansion for initial objects (lemma 3.1.4.2), which gives us the adjoint complement of an identity morphism,

$$\mathrm{id}(0) \quad = \quad \text{¡}(0)$$

expresses the local expansion of an identity derivation:

$$\bot \qquad \overset{\bot<}{\longmapsto} \qquad \dfrac{\bot \qquad \overline{\star}}{\bot}\ \bot{-}^{\dagger}$$

Post-composing with a cobang (lemma 3.1.4.4), which is naturality of the hom set bijection in its codomain coordinate:

$$\text{¡} \cdot i \quad = \quad \text{¡}$$

allows us to take any derivation post-composed with a $\bot{-}^{\dagger}$ rule, and "throw it away" by moving its !-image – i.e. $\mathrm{id}(\star)$ – to the singleton category. This is the permutation conversion for falsehood:

$$\dfrac{\dfrac{\dfrac{\bot \qquad \overline{\star}}{\mathrm{A}}\ \bot{-}^{\dagger}}{\mathcal{E}}}{\mathrm{B}} \qquad \overset{\bot\rightleftarrows}{\longmapsto} \qquad \dfrac{\bot \qquad \overline{\star}}{\mathrm{B}}\ \bot{-}^{\dagger}$$

4.3.4 Context Distributivity Revisited

The context distributivity laws for the positively-presented connectives, the *distributive law* and *absorption law*, are actually both instances of a more general result about cartesian closed categories and adjoint functors called **Frobenius reciprocity**.

Proposition 4.3.4.1 (Frobenius reciprocity) For anti-parallel functors $\mathrm{F} : \mathbb{C} \longrightarrow \mathbb{D}$ and $\mathrm{G} : \mathbb{D} \longrightarrow \mathbb{C}$ between cartesian closed categories, if there is an adjunction $\mathrm{F} \dashv \mathrm{G}$ and the right adjoint G preserves exponentials then for objects $\mathrm{A} : \mathbb{C}$ and $\mathrm{X} : \mathbb{D}$,

$$\mathrm{X} \times \mathrm{F}(\mathrm{A}) \quad \cong \quad \mathrm{F}(\mathrm{G}(\mathrm{X}) \times \mathrm{A})$$

The idea is that X is the interpretation of some ambient context and F is the functor determining some positively-presented connective. Admittedly, the condition that G preserves exponentials seems unmotivated, but when you try to prove the result, you see that it is exactly what is needed to make it go through.

Proof. By the *Yoneda principle*, for an arbitrary $Z : \mathbb{D}$,

$$\begin{array}{ll}
 & \mathbb{D}\,(X \times F(A) \to Z) \\
\cong & \text{[product symmetry]} \\
 & \mathbb{D}\,(F(A) \times X \to Z) \\
\cong & \text{[currying]} \\
 & \mathbb{D}\,(F(A) \to X \supset Z) \\
\cong & \text{[adjoint complement } -^{\flat}\text{]} \\
 & \mathbb{C}\,(A \to G(X \supset Z)) \\
\cong & \text{[assumption that G preserves exponentials]} \\
 & \mathbb{C}\,(A \to G(X) \supset G(Z)) \\
\cong & \text{[uncurrying]} \\
 & \mathbb{C}\,(A \times G(X) \to G(Z)) \\
\cong & \text{[adjoint complement } -^{\sharp}\text{]} \\
 & \mathbb{D}\,(F(A \times G(X)) \to Z) \\
\cong & \text{[product symmetry]} \\
 & \mathbb{D}\,(F(G(X) \times A) \to Z)
\end{array}$$

□

For example, in the case of the adjunction $- + - \dashv \Delta$, Frobenius reciprocity tells us:

$$X \times (+(A\,,B)) \quad \cong \quad +(\Delta(X) \times (A\,,B))$$

or, in other words:

$$X \times (A + B) \quad \cong \quad (X \times A) + (X \times B)$$

which is the *distributive law*!

What is really going on here is that there is a *natural isomorphism* in the *functor category* $(\mathbb{D} \times \mathbb{C}) \supset \mathbb{D}$,

$$\underline{1} \times F(\underline{2}) \quad \cong \quad F(G(\underline{1}) \times \underline{2})$$

Chapter 5

Dependency

5.1 Indexed Categories

5.1.1 Set Indexed Set Families

Dependency is a phenomenon that occurs throughout mathematics and computer science. One example is full *dependent type theory* (about which you are learning in a parallel course). But perhaps more familiar is the idea of an indexed family of sets.

In our discussion of the *axiom of choice* we saw how an ordinary function (in SET) can be used to create a set-indexed family of sets, or **indexed set**: if E and B are sets then a function $f : \mathrm{E} \longrightarrow \mathrm{B}$ gives rise to a function $f^* : \mathrm{B} \longrightarrow \wp(\mathrm{E})$, which we can think of as $f^* : \mathrm{B} \longrightarrow \textsc{Set}$ by the inclusion $\wp(\mathrm{E}) \hookrightarrow \textsc{Set}$.

5.1.2 Catgory Indexed Set Families

We can generalize this setup by using an arbitrary category $\mathbb{C}$ rather than a *discrete* one B to index a family of sets.

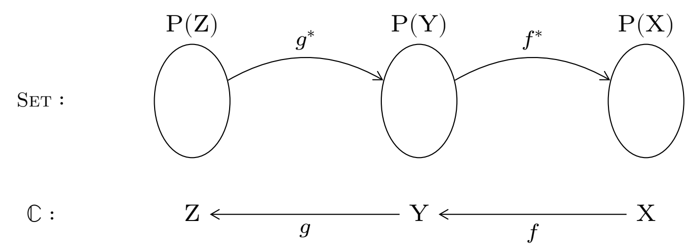
We can define a *contravariant functor* $\mathrm{P} : \mathbb{C}^{\circ} \longrightarrow \textsc{Set}$. The contravariance is not essential (after all, every category is the *opposite* of some other category) but, as we'll soon see, gives the right variance for our intended use of this construction.

Definition 5.1.2.1 (presheaf) A **presheaf**[1] is a contravariant functor from a category to the category of sets:

$$\mathrm{P} : \mathbb{C}^{\circ} \longrightarrow \textsc{Set}$$

[1] The name "presheaf" belies the importance that such functors play in category theory, which is a shame.

We can think of a presheaf as a family of sets indexed by a category. For each object $X : \mathbb{C}$ we have the set $P(X)$ and for each arrow $f : \mathbb{C}(X \to Y)$ we have a function $P(f) : P(Y) \longrightarrow P(X)$. When the presheaf P is known or irrelevant, it is customary to write "f^*" for $P(f)$.



Perhaps the most important kind of presheaf is a *representable presheaf*, which, recall, is the contravariant functor represented by an object of $\mathbb{C}$:

$$\mathbb{C}(- \to Z) \quad : \quad \mathbb{C}^\circ \longrightarrow \textsc{Set}$$

In this case, the set corresponding to each object in $\mathbb{C}$ is the set of $\mathbb{C}$-arrows from that object to the representative object.

5.1.3 Catgory Indexed Category Families

We can generalize this setup even more: instead of a family of sets indexed by a category, we can create a family of categories indexed by a category.

Definition 5.1.3.1 (indexed category) An **indexed category** is a contravariant functor from a category to the 2-category of (small) categories:

$$P : \mathbb{C}^\circ \longrightarrow \textsc{Cat}$$

The category $\mathbb{C}$ is called the **base category** of the indexed category P, for an object $X : \mathbb{C}$, the category $P(X)$ is called the **fiber** of P over X, and for arrow $f : \mathbb{C}(X \to Y)$ the functor $f^* : P(Y) \longrightarrow P(X)$ is called the **reindexing functor** induced by f.

We can also do a version of this construction in which the codomain of the functor P is an arbitrary 2-category of categories, not necessarily Cat. In particular, we will be interested in **indexed bicartesian closed categories**, which are functors $P : \mathbb{C}^\circ \longrightarrow$ BCC.

5.2 Interpretation of Predicate Logic

5.2.1 Interpretation of Terms

In order to move beyond propositional logic, we need to be able to interpret predicates over a term language. Our term language will be typed, but because our focus is the logic, we will not introduce any type formers. We explain the interpretation of the term language only briefly, as it is not the focus of this section.

We begin with an arbitrary collection of **atomic types** and interpret them as objects of a cartesian category $\mathbb{C}$. For type X,

$$[\![\mathrm{X}]\!] \;:\; \mathbb{C}$$

A **typing context** Φ is a list of distinct typed variables:

$$\Phi \quad = \quad x_1 : \mathrm{X}_1 \;,\; \cdots \;,\; x_n : \mathrm{X}_n$$

We use the cartesian structure of $\mathbb{C}$ to interpret typing contexts in the same way we did for propositional contexts:

$$\begin{aligned} [\![\emptyset]\!] \quad &:= \quad 1 \\ [\![\Phi \,, x : \mathrm{X}]\!] \quad &:= \quad [\![\Phi]\!] \times [\![\mathrm{X}]\!] \end{aligned}$$

The idea is that the names of the variables are immaterial, they merely act as *projections* out of a product.

Weakening of typing contexts lets us forget about some of the variables in scope. by the structural rule of context exchange, it suffices to consider just the case of a **single omission**:

$$\hat{x} \quad : \quad \Phi \,, x : \mathrm{X} \longmapsto \Phi$$

Its interpretation is the same as that of propositional *context weakening*, namely a *projection*, or up to isomorphism,

$$[\![\hat{x}]\!] \quad := \quad \mathrm{id} \times ! \quad : \quad [\![\Phi]\!] \times [\![\mathrm{X}]\!] \longrightarrow [\![\Phi]\!]$$

Next, we consider an arbitrary collection of typed **function symbols**, each of the form,

$$f \quad \in \quad \mathcal{F}(\mathrm{Y}_1 \,, \cdots , \mathrm{Y}_n \,; \mathrm{X})$$

where X and each Y_i is a type. The idea is that when the function symbol f is applied to terms of types $\vec{\mathrm{Y}}$ the result will be a term of type X. We follow the usual convention that a constant symbol c of type X is a nullary function symbol,

$$c \quad \in \quad \mathcal{F}(\emptyset \,; \mathrm{X})$$

We interpret function symbols as morphisms from the product of the interpretations of their argument types to that of their result type:

$$[\![f \in \mathcal{F}(\mathrm{Y}_1 , \cdots , \mathrm{Y}_n \,; \mathrm{X})]\!] \quad : \quad [\![\mathrm{Y}_1]\!] \times \cdots \times [\![\mathrm{Y}_n]\!] \longrightarrow [\![\mathrm{X}]\!]$$

Open terms, or **terms in context** are built inductively from the function symbols. We express that term t has type X in typing context Φ as "$\Phi \mid t : \mathrm{X}$". In order for such a term in context to be valid, the context Φ must contain at least the typed free variables that occur in t. However, the context Φ may also contain additional **dummy variables** that do not occur free in t.

Our inductive **interpretation of terms** follows their inductive construction from the function symbols:

lifted variable: for variable $x_i \in \Phi$,

$$[\![\Phi \mid x_i : \mathrm{X}_i]\!] \quad := \quad [\![\Phi]\!] \xrightarrow{\quad \pi_i \quad} [\![\mathrm{X}]\!]$$

applied function symbol: for function symbol $f \in \mathcal{F}(\mathrm{Y}_1 , \cdots , \mathrm{Y}_n \,; \mathrm{X})$ and terms $\Phi \mid t_1 : \mathrm{Y}_1 , \cdots , t_n : \mathrm{Y}_n$,

$$[\![\Phi \mid f(t_1,\cdots,t_n) : \mathrm{X}]\!] \quad := \quad [\![\Phi]\!] \xrightarrow{\langle [\![t_1]\!] , \cdots , [\![t_n]\!] \rangle} [\![\mathrm{Y}_1]\!] \times \cdots \times [\![\mathrm{Y}_n]\!] \xrightarrow{\quad [\![f]\!] \quad} [\![\mathrm{X}]\!]$$

context extension: for term $\Phi \mid t : \mathrm{X}$ and variable $y \notin \Phi$,

$$[\![\Phi , y : \mathrm{Y} \mid t : \mathrm{X}]\!] \quad := \quad [\![\Phi]\!] \times [\![\mathrm{Y}]\!] \xrightarrow{\quad [\![\hat{y}]\!] \quad} [\![\Phi]\!] \xrightarrow{\quad [\![t]\!] \quad} [\![\mathrm{X}]\!]$$

substitution: for terms $\Phi , y : \mathrm{Y} \mid t : \mathrm{X}$ and $\Phi \mid s : \mathrm{Y}$,

$$[\![\Phi \mid t[y \mapsto s] : \mathrm{X}]\!] \quad := \quad [\![\Phi]\!] \xrightarrow{\langle [\![\Phi]\!] , [\![s]\!] \rangle} [\![\Phi]\!] \times [\![\mathrm{Y}]\!] \xrightarrow{\quad [\![t]\!] \quad} [\![\mathrm{X}]\!]$$

A substitution of just one term for a variable is a **single substitution**. We call the precomposed arrow in the definition of a term under single substitution the interpretation of the single substitution:

$$[\![\, [y \mapsto s] \,]\!] \quad := \quad \langle \mathrm{id} , [\![s]\!] \rangle$$

The reason that it tuples the interpretation of the substituting term with that of its context is that in the language, applying such a substitution to a term leaves undisturbed any other variables that may occur in the term.

5.2.2 Interpreting of Predicates

Interpreting Propositions

In order to define predicates we begin with a collection of typed **relation symbols**, each of the form,

$$\mathrm{R} \quad \in \quad \mathcal{R}(\mathrm{X}_1\,,\cdots,\mathrm{X}_n)$$

where each X_i is a type. The idea is that when the relation symbol R is applied to terms of types $\vec{\mathrm{X}}$ the result is an atomic proposition, or **predicate**.

We will need to interpret not only closed predicates (those without free term variables), but open ones as well. Thus predicates are *dependent* on their free term variables. This suggests that we interpret predicates in an *indexed category* over a base category interpreting their typing contexts and terms.

Given an interpretation of the term language in a cartesian category $\mathbb{C}$, we interpret relation symbols in a $\mathbb{C}$-indexed bicartesian closed category $\mathrm{P} : \mathbb{C}^\circ \longrightarrow$ BCC as objects in the fibers over the product of the interpretations of their argument types:

$$[\![\mathrm{R} \in \mathcal{R}(\mathrm{X}_1\,,\cdots,\mathrm{X}_n)]\!] \quad : \quad \mathrm{P}([\![\mathrm{X}_1]\!] \times \cdots \times [\![\mathrm{X}_n]\!])$$

As in the case of terms, we need to keep track of the typing context in which a predicate (and in general, a proposition, which may contain connectives) occurs. Thus we speak of **propositions in context** and write "$\Phi \mid \mathrm{A}$ PROP" to indicate that A is a proposition, all of whose free variables are contained in the typing context Φ.

Our inductive **interpretation of predicates** follows their inductive construction from the relation symbols:

applied relation symbol: for relation symbol $\mathrm{R} \in \mathcal{R}(\mathrm{Y}_1\,,\cdots,\mathrm{Y}_n)$ and terms $\Phi \mid t_i : \mathrm{Y}_i$,

$$[\![\Phi \mid \mathrm{R}(t_1\,,\cdots,t_n)\ \textsc{prop}]\!] \quad := \quad \langle [\![t_1]\!]\,,\cdots,[\![t_n]\!]\rangle^*\ ([\![\mathrm{R}]\!]) \ : \ \mathrm{P}([\![\Phi]\!])$$

context extension: for predicate $\Phi \mid \mathrm{A}$ PROP and variable $y \notin \Phi$,

$$[\![\Phi\,, y : \mathrm{Y} \mid \mathrm{A}\ \textsc{prop}]\!] \quad := \quad [\![\hat{y}]\!]^*\ ([\![\mathrm{A}]\!]) \ : \ \mathrm{P}([\![\Phi\,, y : \mathrm{Y}]\!])$$

substitution: for predicate $\Phi\,, y : \mathrm{Y} \mid \mathrm{A}$ PROP and term $\Phi \mid s : \mathrm{Y}$,

$$[\![\Phi \mid \mathrm{A}[y{\mapsto}s]\ \textsc{prop}]\!] \quad := \quad [\![\,[y{\mapsto}s]\,]\!]^*\ ([\![\mathrm{A}]\!]) \ : \ \mathrm{P}([\![\Phi]\!])$$

Note that in each case we simply *reindex* by the same morphism with which we precomposed in the inductive interpretation of terms.

Finally, in order to interpret full predicate logic, we use the interpretations of the connectives that we presented earlier.

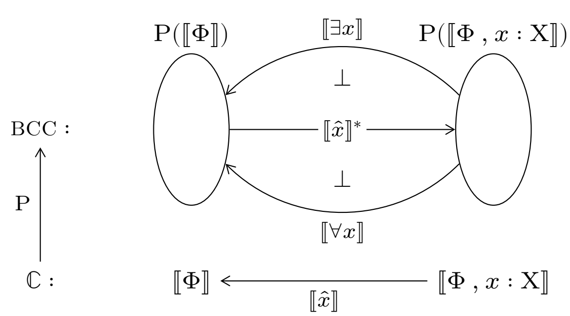
Remark 5.2.2.1 (strictness) Because our interpretation is in an indexed *bicartesian closed category*, we know that the reindexing functors will respect the bicartesian closed structure of the fibers (e.g. send products to products) This sounds nice and tidy, but there is a subtle issue having to do with *strictness*.

Consider a term $\Phi \mid t : \mathrm{X}$ and proposition $\Phi\, , x : \mathrm{X} \mid \mathrm{A} \wedge \mathrm{B}$ Prop. The object $[\![\, [x{\mapsto}t]\,]\!]^*([\![\mathrm{A} \wedge \mathrm{B}]\!])$ will indeed be *a* cartesian product of $[\![\, [x{\mapsto}t]\,]\!]^*([\![\mathrm{A}]\!])$ and $[\![\, [x{\mapsto}t]\,]\!]^*([\![\mathrm{B}]\!])$ in $\mathrm{P}([\![\Phi]\!])$, but there is no *a priori* reason to assume that it will be *the same* cartesian product as $[\![\mathrm{A}[x{\mapsto}t] \wedge \mathrm{B}[x{\mapsto}t]]\!]$ even though $(\mathrm{A} \wedge \mathrm{B})[x{\mapsto}t]$ and $\mathrm{A}[x{\mapsto}t] \wedge \mathrm{B}[x{\mapsto}t]$ are usually considered to be the same proposition in the language.

This is an issue known as *soundness for syntax*, and there are established techniques for dealing with it, but we will not have time to consider them in this course.

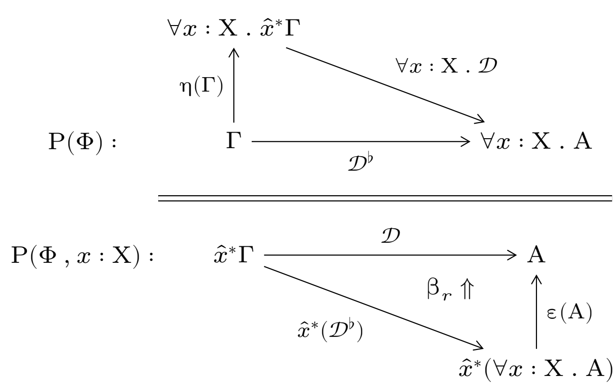
5.3 Interpretation of Quantification

Bill Lawvere observed that generalized quantifiers could be interpreted as adjoints to reindexing functors, which led to his *hyperdoctrine interpretation of first-order logic*. In the familiar case, where we quantify only over *bound variables* rather than more general *bound terms*, the interpretations of the quantifiers $\forall x$ and $\exists x$ are respectively right and left adjoints to the reindexing functors determined by the context weakening $\hat{x}$:



5.3.1 Universal Quantification

Suppressing the semantic brackets for readability, the internal picture of this adjunction looks like this:



In this adjunction, the adjoint complement operation $-^\flat$ takes a derivation from assumptions in which a given variable does not occur free to a derivation of the universal quantification over that variable of the goal:

$$\mathcal{D} \;:\; \hat{x}^*\Gamma \longrightarrow \mathrm{A} \qquad \overset{-^\flat}{\longmapsto} \qquad \mathcal{D}^\flat \;:\; \Gamma \longrightarrow \forall x : \mathrm{X} \,.\, \mathrm{A}$$

We can express this with the following inference rule of natural deduction:

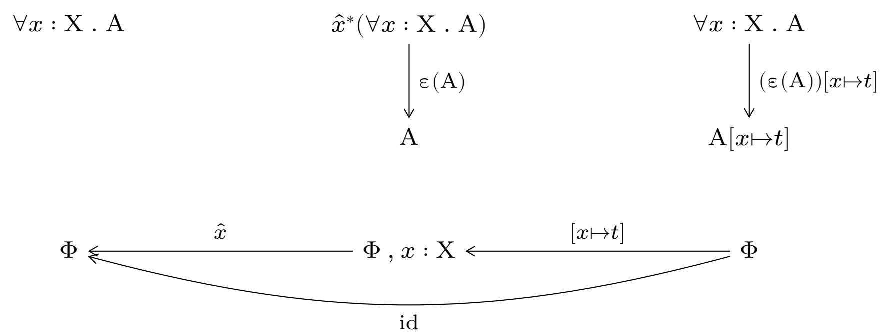
$$\dfrac{\dfrac{\hat{x}^*\Gamma}{\mathcal{D}}}{\mathrm{A}} \qquad \overset{-^\flat}{\longmapsto} \qquad \dfrac{\Gamma \qquad \dfrac{\dfrac{\overline{\hat{x}^*\Gamma}}{\mathcal{D}}}{\mathrm{A}}}{\forall x : \mathrm{X} \,.\, \mathrm{A}}\ \forall{+}^\dagger$$

This is our derived introduction rule for universal quantification. It is equivalent to the Gentzen rule with the context variable x acting as the private variable of the hypothetical subderivation. Observe that the side condition of the Gentzen rule is automatically enforced because the rest of the derivation is in the category P(Φ), where the variable x is not in scope.

The counit of this adjunction when written as an inference rule becomes our derived elimination rule:

$$\mathrm{A} \qquad \overset{\varepsilon}{\longmapsto} \qquad \dfrac{\hat{x}^*(\forall x : \mathrm{X} \,.\, \mathrm{A})}{\mathrm{A}}\ \forall{-}^\dagger$$

The corresponding Gentzen rule allows us to conclude any instance of the formula A in which a type-appropriate term t is substituted for the variable x and any remaining free variables of A are left undisturbed. Categorically, this corresponds to reindexing by the interpretation of the *single substitution* induced by the term in context $\Phi \mid t : X$. Thus, the Gentzen elimination rule corresponds to $(\varepsilon(A))[x{\mapsto}t]$, as shown:



The Gentzen rule gives us a stronger result by yielding a derivation in the original typing context Φ. But it also makes a stronger demand by requiring us to immediately choose a term of type X in that context to act as the **representative** of its type under consideration. The adjoint-theoretic rule takes a more relaxed approach. It gives us the option, but not the obligation, of supplying a representative term at some point in the future, while in the meantime choosing the context variable x to act as a **generic representative**.

To wit, the context variable x *is* a term of type X, but in the extended context $\Phi , x : X$, and *not* in the original context Φ. At any point we may choose a representative term $\Phi \mid t : X$ to use in place of this context variable by performing the substitution $[x{\mapsto}t]$, but until we do so our derivation is valid only in the extended context, a world in which a term of the specified type exists by fiat, in the form of a context variable. The derived rule has the advantage that we need not decide *which* term t to substitute for x in A at the time we apply the rule. This has useful applications to the study of metavariables and proof search.

The commuting triangle of the universal property of the counit,

$$\widehat{x}^*(\mathcal{D}^\flat) \cdot \varepsilon(A) \quad \overset{\beta_r}{=} \quad \mathcal{D}$$

expresses the local reduction for universal quantification in the category $P(\Phi , x : X)$:

$$\cfrac{\cfrac{\widehat{x}^*\Gamma \qquad \cfrac{\cfrac{\overline{\widehat{x}^*\Gamma}}{\mathcal{D}}}{A}}{\widehat{x}^*(\forall x : X . A)}\ \widehat{x}^*(\forall +^\dagger)}{A}\ \forall -^\dagger \qquad \overset{\forall >}{\longmapsto} \qquad \cfrac{\widehat{x}^*\Gamma}{\cfrac{\mathcal{D}}{A}}$$

We recover the Gentzen version by applying a single substitution for x to both sides.

The equation for the adjoint complement of a counit component,

$$\mathrm{id}(\forall x : \mathrm{X}\,.\,-) \quad = \quad (\varepsilon\ -)^\flat$$

expresses the local expansion of an identity derivation:

$$\forall x : \mathrm{X}\,.\,\mathrm{A} \quad \overset{\forall<}{\longmapsto} \quad \dfrac{\forall x : \mathrm{X}\,.\,\mathrm{A} \qquad \dfrac{\overline{\hat{x}^*(\forall x : \mathrm{X}\,.\,\mathrm{A})}}{\mathrm{A}}\ \forall-^\dagger}{\forall x : \mathrm{X}\,.\,\mathrm{A}}\ \forall+^\dagger$$

We get the local expansion for an arbitrary derivation by precomposing it and applying the permutation conversion, which is the naturality of the hom set bijection of this adjunction in the domain coordinate,

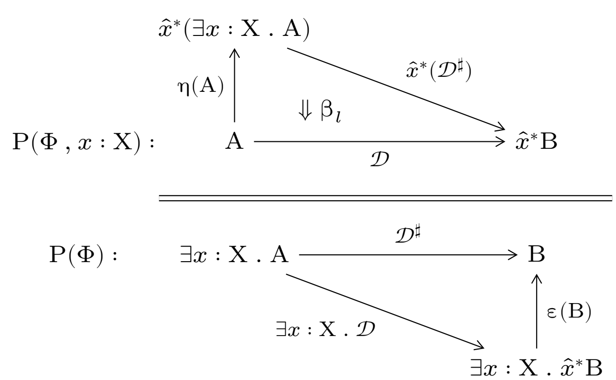
$$\mathcal{E} \cdot \mathcal{D}^\flat \quad = \quad (\hat{x}^*\mathcal{E} \cdot \mathcal{D})^\flat$$

or,

$$\dfrac{\dfrac{\dfrac{\Gamma}{\mathcal{E}}}{\mathrm{C}} \qquad \dfrac{\dfrac{\overline{\hat{x}^*\mathrm{C}}}{\mathcal{D}}}{\mathrm{A}}}{\forall x : \mathrm{X}\,.\,\mathrm{A}}\ \forall+^\dagger \quad \overset{\forall\rightleftarrows}{\longmapsto} \quad \dfrac{\Gamma \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\hat{x}^*\Gamma}}{\hat{x}^*\mathcal{E}}}{\hat{x}^*\mathrm{C}}}{\mathcal{D}}}{\mathrm{A}}}{\forall x : \mathrm{X}\,.\,\mathrm{A}}\ \forall+^\dagger$$

5.3.2 Existential Quantification

Again suppressing the semantic brackets for readability, the internal picture of this adjunction looks like this:



In this adjunction, the adjoint complement operation $-^\sharp$ takes a derivation with a goal in which a given variable does not occur free to a derivation from the existential quantification over that variable of the assumption:

$$\mathcal{D} \;:\; \mathrm{A} \longrightarrow \hat{x}^*\mathrm{B} \qquad \overset{-^\sharp}{\longmapsto} \qquad \mathcal{D}^\sharp \;:\; \exists x : \mathrm{X} \,.\, \mathrm{A} \longrightarrow \mathrm{B}$$

We can express this with the following inference rule of natural deduction:

$$\dfrac{\dfrac{\mathrm{A}}{\mathcal{D}}}{\hat{x}^*\mathrm{B}} \qquad \overset{-^\sharp}{\longmapsto} \qquad \dfrac{\exists x : \mathrm{X} \,.\, \mathrm{A} \qquad \dfrac{\dfrac{\overline{\mathrm{A}}}{\mathcal{D}}}{\hat{x}^*\mathrm{B}}}{\mathrm{B}} \; \exists{-}^\dagger$$

This is our derived elimination rule for existential quantification. As with the derived rule for universal introduction, the context variable x plays the role of the private variable and Gentzen's side condition is automatically enforced because the rest of the derivation is in the category $\mathrm{P}(\Phi)$, where x is not in scope.

As with the other positively presented connectives (falsehood and disjunction), this rule does not directly encode compatibility with ambient propositional contexts. For that, we need the instance of *Frobenius reciprocity*,

$$frob \quad : \quad \Gamma \times (\exists x : \mathrm{X} \,.\, \mathrm{A}) \;\cong\; \exists x : \mathrm{X} \,.\, \hat{x}^*(\Gamma) \times \mathrm{A}$$

Indeed, this is the motivating case for that concept. Precomposing $frob$ with the derived rule yields Gentzen's rule:

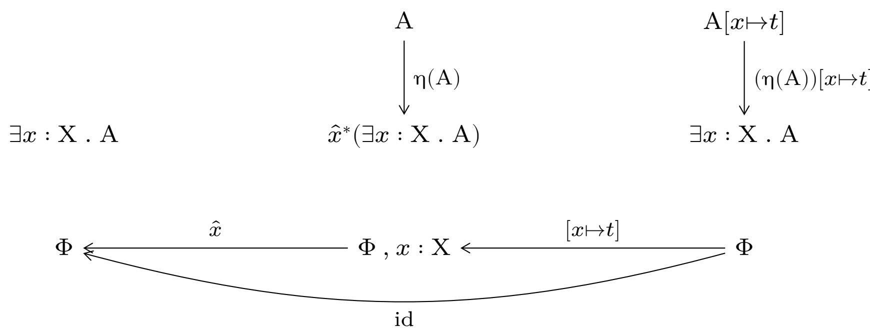
FIXME

$$\dfrac{\dfrac{\Gamma \qquad \dfrac{\dfrac{\Gamma}{\mathcal{E}}}{\exists x : \mathrm{X} \,.\, \mathrm{A}}}{\exists x : \mathrm{X} \,.\, \hat{x}^*\Gamma \wedge \mathrm{A}}\, frob \qquad \dfrac{\dfrac{\overline{\hat{x}^*\Gamma} \qquad \overline{\mathrm{A}}}{\mathcal{D}}}{\hat{x}^*\mathrm{B}}}{\mathrm{B}} \; \exists{-}^\dagger$$

The unit of this adjunction when written as an inference rule becomes our derived introduction rule:

$$\mathrm{A} \qquad \overset{\eta}{\longmapsto} \qquad \dfrac{\mathrm{A}}{\hat{x}^*(\exists x : \mathrm{X} \,.\, \mathrm{A})} \; \exists{+}^\dagger$$

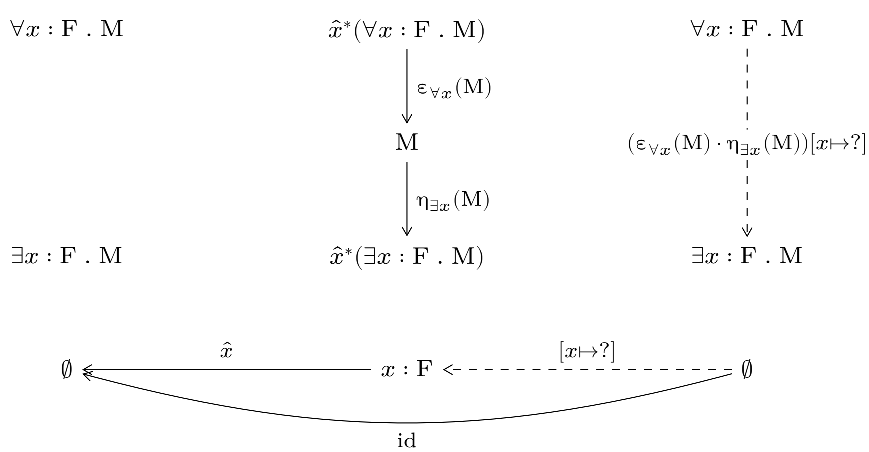
Dual to the case of $\forall{-}^\dagger$, we recover the Gentzen rule by reindexing along a *single substitution*:



Like $\forall -^{\dagger}$, this rule is more lenient than Gentzen's because it does not require us to immediately produce a **witness** at the time we apply the rule, but rather allows us to temporarily use a context variable as a **generic witness**. This has the effect of leaving x as a free variable (i.e. metavariable or logic variable) to be instantiated later, for example through *unification*. But it also means that in order to obtain a derivation in the world in which we started, the typing context Φ, we must eventually produce a witness term $\Phi \mid t : \mathrm{X}$ to substitute for the context variable x. Otherwise, we have a derivation that is valid only in a world where such a term is assumed to exist.

The importance of knowing which fiber we are in is illustrated by the attempt to prove the existence of a magical fairy from the well-known fact that all fairies are magical:

Example 5.3.2.1 (magical fairies) If we allow F to represent the type of fairies and M the predicate that x is magical, then we may derive $x : \mathrm{F} \mid \forall x : \mathrm{F} \,.\, \mathrm{M} \vdash \exists x : \mathrm{F} \,.\, \mathrm{M}$ as shown.



Then the fulfillment of childhood dreams awaits us in our own world – just as soon as we are able to produce any fairy whatsoever.

The commuting triangle of the universal property of the unit,

$$\eta(\mathrm{A}) \cdot \hat{x}^*(\mathcal{D}^\sharp) \quad \overset{\beta_\iota}{=} \quad \mathcal{D}$$

expresses the local reduction for existential quantification in the category $\mathrm{P}(\Phi\,, x : \mathrm{X})$:

$$\dfrac{\dfrac{\mathrm{A}}{\hat{x}^*(\exists x : \mathrm{X}\,.\,\mathrm{A})}\;\exists{+}^\dagger \qquad \dfrac{\dfrac{\overline{\mathrm{A}}}{\mathcal{D}}}{\hat{x}^*\mathrm{B}}}{\hat{x}^*\mathrm{B}}\;\hat{x}^*(\exists{-}^\dagger) \qquad \overset{\exists>}{\longmapsto} \qquad \dfrac{\dfrac{\mathrm{A}}{\mathcal{D}}}{\hat{x}^*\mathrm{B}}$$

We recover the Gentzen version by precomposing $frob$ and applying a single substitution for x to both sides.

The equation for the adjoint complement of a unit component,

$$\mathrm{id}(\exists x : \mathrm{X}\,.\,-) \quad = \quad (\eta\ -)^\sharp$$

expresses the local expansion of an identity derivation:

$$\exists x : \mathrm{X}\,.\,\mathrm{A} \qquad \overset{\exists<}{\longmapsto} \qquad \dfrac{\exists x : \mathrm{X}\,.\,\mathrm{A} \qquad \dfrac{\overline{\mathrm{A}}}{\hat{x}^*(\exists x : \mathrm{X}\,.\,\mathrm{A})}\;\exists{+}^\dagger}{\exists x : \mathrm{X}\,.\,\mathrm{A}}\;\exists{-}^\dagger$$

The naturality of the hom set bijection of this adjunction in the codomain coordinate,

$$\mathcal{D}^\sharp \cdot \mathcal{E} \quad = \quad (\mathcal{D} \cdot \hat{x}^*\mathcal{E})^\sharp$$

or

$$\dfrac{\dfrac{\exists x : \mathrm{X}\,.\,\mathrm{A} \qquad \dfrac{\dfrac{[\mathrm{A}]}{\mathcal{D}}}{\hat{x}^*\mathrm{B}}}{\mathrm{B}}\;\exists{-}^\dagger}{\dfrac{\mathcal{E}}{\mathrm{C}}} \qquad \overset{\exists\rightleftarrows}{\longmapsto} \qquad \dfrac{\exists x : \mathrm{X}\,.\,\mathrm{A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\mathrm{A}]}{\mathcal{D}}}{\hat{x}^*\mathrm{B}}}{\hat{x}^*\mathcal{E}}}{\hat{x}^*\mathrm{C}}}{\mathrm{C}}\;\exists{-}^\dagger$$

is the permutation conversion for existential quantification.

5.3.3 Substitution Compatibility

In logic, the quantifiers–like the propositional connectives–are compatible with (capture-avoiding) substitution:

$$\begin{array}{rcl} (\forall x : \mathrm{X}\,.\,\mathrm{A})[y{\mapsto}t] & = & \forall x : \mathrm{X}\,.\,(\mathrm{A}[y{\mapsto}t]) \\ (\exists x : \mathrm{X}\,.\,\mathrm{A})[y{\mapsto}t] & = & \exists x : \mathrm{X}\,.\,(\mathrm{A}[y{\mapsto}t]) \end{array}$$

We need to ensure that this property holds in our categorical semantics. Unlike the case of context distributivity, there is nothing special about bicartesian closed categories that guarantees this will be the case.

First, observe that the interpretation of a *single substitution* commutes with that of a *single omission*:

$$\mathbb{C}: \quad \begin{array}{ccc} [\![\Phi , x : \mathrm{X}]\!] & \xrightarrow{[\![[y \mapsto t]]\!]} & [\![\Phi , x : \mathrm{X} , y : \mathrm{Y}]\!] \\ \big\downarrow{\scriptstyle [\![\hat{x}]\!]} & & \big\downarrow{\scriptstyle [\![\hat{x}]\!]} \\ [\![\Phi]\!] & \xrightarrow[{[\![[y \mapsto t]]\!]}]{} & [\![\Phi , y : \mathrm{Y}]\!] \end{array}$$

This is a consequence of the fact that the interpretation of a single substitution in an extended context is the product of the single substitution and the context extension:

$$\begin{array}{ccc} & [\![\Phi , y : \mathrm{Y} , x : \mathrm{X}]\!] & \\ {\scriptstyle [\![[y \mapsto t]]\!] \times [\![\mathrm{X}]\!]} \nearrow & & \searrow {\scriptstyle [\![\Phi]\!] \times \sigma} \\ [\![\Phi , x : \mathrm{X}]\!] & \xrightarrow[{[\![[y \mapsto t]]\!]}]{} & [\![\Phi , x : \mathrm{X} , y : \mathrm{Y}]\!] \end{array}$$

together with the definition of a *product of arrows*.

This gives us a commuting square of the corresponding reindexing functors in BCC:

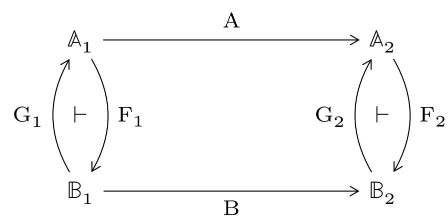
$$\mathrm{BCC}: \quad \begin{array}{ccc} \mathrm{P}([\![\Phi , x : \mathrm{X}]\!]) & \xleftarrow{[\![[y \mapsto t]]\!]^*} & \mathrm{P}([\![\Phi , x : \mathrm{X} , y : \mathrm{Y}]\!]) \\ \big\uparrow{\scriptstyle [\![\hat{x}]\!]^*} & & \big\uparrow{\scriptstyle [\![\hat{x}]\!]^*} \\ \mathrm{P}([\![\Phi]\!]) & \xleftarrow[{[\![[y \mapsto t]]\!]^*}]{} & \mathrm{P}([\![\Phi , y : \mathrm{Y}]\!]) \end{array} \tag{5.1}$$

Recall that $[\![\exists x]\!]$ and $[\![\forall x]\!]$ are adjoints to $[\![\hat{x}]\!]^*$. So it would suffice if we had a condition that guaranteed that whenever the square above commutes, for $\mho \in \{\forall , \exists\}$, this one does as well:

$$\mathrm{BCC}: \quad \begin{array}{ccc} \mathrm{P}([\![\Phi , x : \mathrm{X}]\!]) & \xleftarrow{[\![[y \mapsto t]]\!]^*} & \mathrm{P}([\![\Phi , x : \mathrm{X} , y : \mathrm{Y}]\!]) \\ \big\downarrow{\scriptstyle [\![\mho x]\!]} & & \big\downarrow{\scriptstyle [\![\mho x]\!]} \\ \mathrm{P}([\![\Phi]\!]) & \xleftarrow[{[\![[y \mapsto t]]\!]^*}]{} & \mathrm{P}([\![\Phi , y : \mathrm{Y}]\!]) \end{array} \tag{5.2}$$

Here's how we can arrange it.

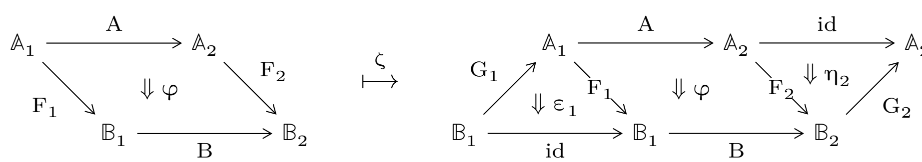
Lemma 5.3.3.1 Given adjunctions $F_1 \dashv G_1$ and $F_2 \dashv G_2$ and functors A and B as shown (without assuming that anything commutes):



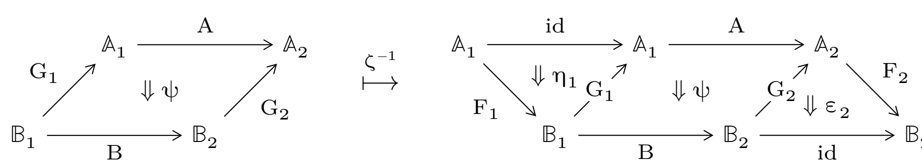
there is a bijection between sets of natural transformations:

$$\frac{\mathbb{A}_1 \supset \mathbb{B}_2\,(\mathrm{A}\cdot \mathrm{F}_2 \to \mathrm{F}_1 \cdot \mathrm{B})}{\mathbb{B}_1 \supset \mathbb{A}_2\,(\mathrm{G}_1 \cdot \mathrm{A} \to \mathrm{B}\cdot \mathrm{G}_2)}\ \zeta$$

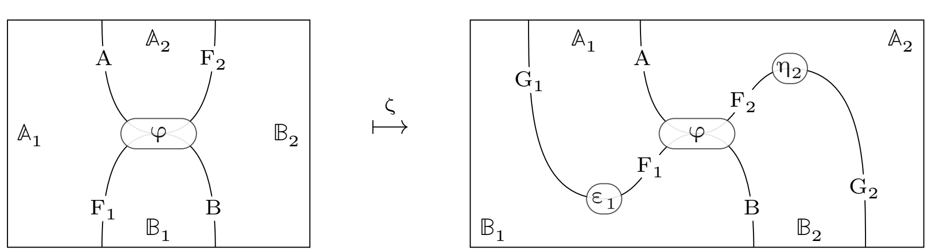
Proof. Such a bijection is given by pasting with the respective unit and counit of the two adjunctions:



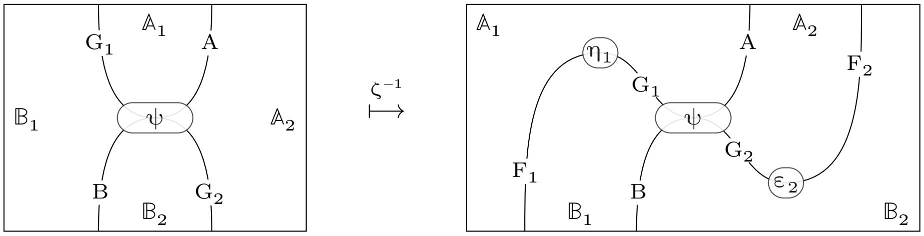
and



or equivalently, as string diagrams:



and



In this form, it's easy to see how the *zigzag laws* ensure that going back and forth in either order amounts to the identity. □

Definition 5.3.3.2 (mate) Natural transformations related by the bijection ζ are each called the other's **mate**. If we need to be more specific, we call φ the left mate of ψ and ψ the right mate of φ, after the adjoint functors involved.

Definition 5.3.3.3 (Beck-Chevalley condition) The **Beck-Chevalley condition** is the requirement that the mate of a *natural isomorphism* is itself a natural isomorphism.

So in order to ensure that all squares of the form 5.2 commute (at least up to isomorphism), it suffices to require that the Beck-Chevaley condition hold in cases where one of the adjoint functors in each adjunction is the reindexing functor determined by a projection. Then the natural isomorphism in square 5.2 will be the mate of the identity in square 5.1, and its component at a proposition A will be the isomorphisms between $(\mho\ x : \mathrm{X}\ .\ \mathrm{A})[y{\mapsto}t]$ and $\mho\ x : \mathrm{X}\ .\ (\mathrm{A}[y{\mapsto}t])$.