A Double-Categorical Perspective on Type Universes

Edward Morehouse G. A. Kavvos Daniel R. Licata

Types 2019

Motivation

The *Univalence* axiom of HoTT says that for types A and B in a universe $\mathcal{U}$:

$$\overbrace{(\mathrm{A} =_{\mathcal{U}} \mathrm{B})}^{\text{paths in } \mathcal{U} \text{ from } \mathrm{A} \text{ to } \mathrm{B}} \quad \simeq \quad \underbrace{(\mathrm{A} \simeq \mathrm{B})}_{\text{functions } \mathrm{A} \to \mathrm{B} \text{ satisfying a predicate}}$$

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Choosing the predicate for (coherent) equivalences yields the bidirectional paths of the *identity type*.

Different choices of function predicate correspond to different path structure on $\mathcal{U}$.

Every path has an underlying function. We *coerce* along a path by applying that function.

Slogan: “all paths are coercible”.

Set Up

We pursue this perspective on Univalence to develop a categorical model for type-universes that is

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We do this using the structure of double categories.

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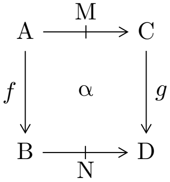

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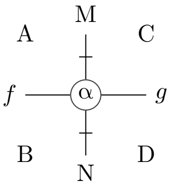

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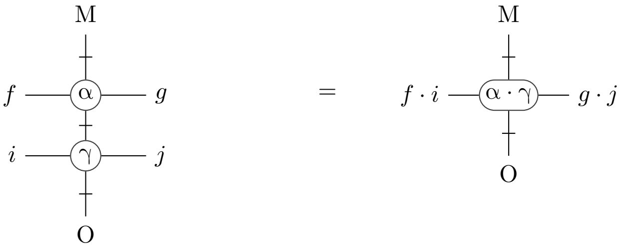

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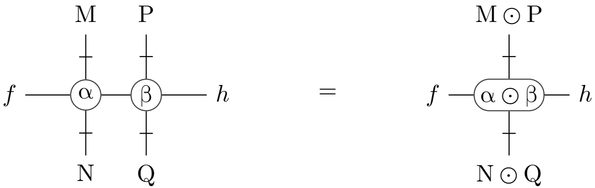
(weak) composition in the proarrow dimension:



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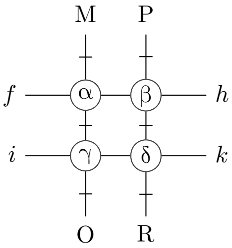

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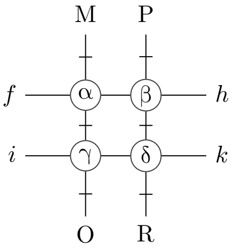

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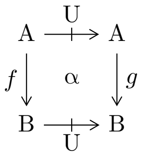
A coherence theorem allows us to ignore and recover coherators [GP99].

Globular Squares

A square with trivial boundary in some dimension is a *globe*.



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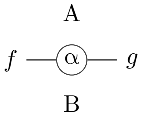

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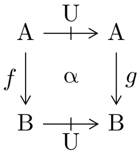

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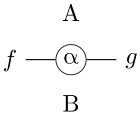

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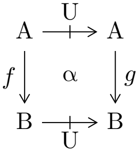

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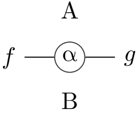

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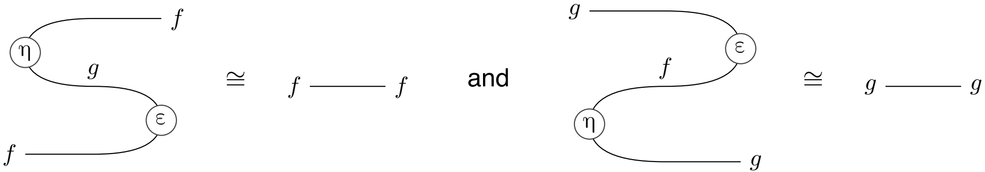
An *arrow adjunction* in a double category is formed by arrows $f : A \to B$ and $g : B \to A$ and arrow globes $\eta : \mathrm{id}(A) \nrightarrow f \cdot g$ and $\varepsilon : g \cdot f \nrightarrow \mathrm{id}(B)$ such that:



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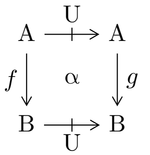

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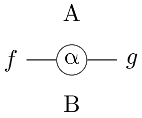

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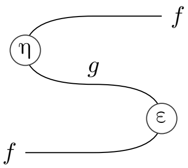

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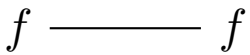

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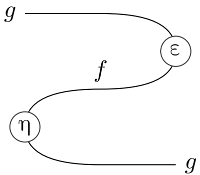

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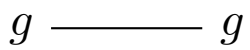

(The "$- \cong -$" accounts for boundary coherators.)

Companion Structure

In a double category, parallel arrow $f : \mathrm{A} \to \mathrm{B}$ and proarrow $\mathrm{M} : \mathrm{A} \nrightarrow \mathrm{B}$ are *companions* [GP04] if there are *connection squares*

$$\begin{array}{ccc} \mathrm{A} & \overset{\mathrm{M}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & \mathrm{B} \\ {\scriptstyle f}\downarrow & {}^{\cdot}f_{\lrcorner} & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{B} & \underset{\mathrm{U}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & \mathrm{B} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{A} & \overset{\mathrm{U}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & \mathrm{A} \\ {\scriptstyle \mathrm{id}}\downarrow & {}^{\ulcorner}f_{.} & \downarrow{\scriptstyle f} \\ \mathrm{A} & \underset{\mathrm{M}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & \mathrm{B} \end{array}$$

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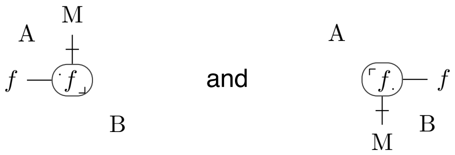

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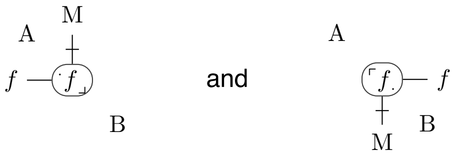
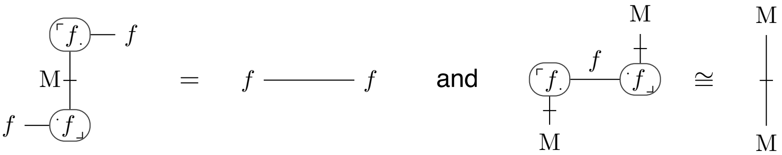


satisfying the *companion laws*:



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When they exist, companion morphisms are unique up to canonical isoglobes.
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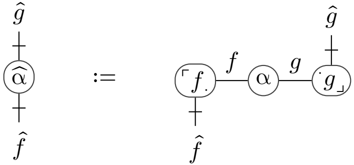

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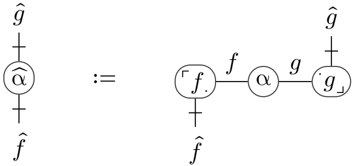

Companionship also respects globe composition structure (contravariantly).
There are $(0, 2)$-full sub-double categories of companionable arrow- and proarrow globes, which are equivalent as bicategories.

Paths as Proarrows

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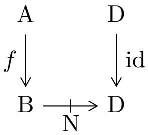
This gives us the following Univalence-like principle for types A and B in a universe $\mathcal{U}$:

$$\overbrace{(\mathrm{A} \rightsquigarrow \mathrm{B})}^{\text{directed paths in } \mathcal{U} \text{ from A to B}} \quad \simeq \quad \underbrace{(f : \mathrm{A} \to \mathrm{B} \mid \mathrm{comp}\, f)}_{\text{companionable functions from A to B}}$$

Kan Structure

Companion structure on a double category interprets a form of Kan structure for the universe.

For arbitrary path and composable companionable function:



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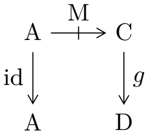

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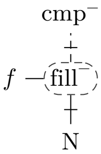

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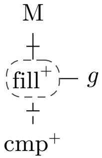

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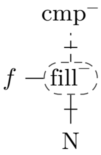

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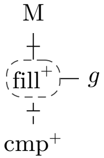

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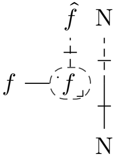

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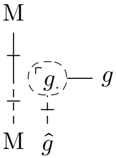

Contravariant Paths

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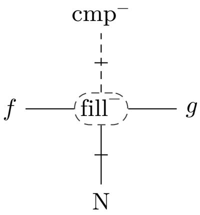

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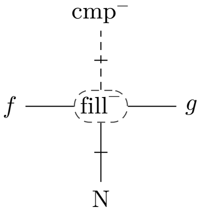

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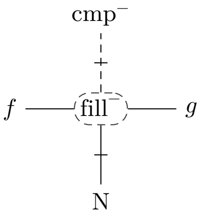

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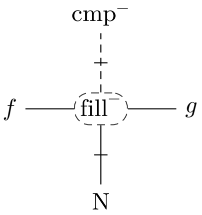

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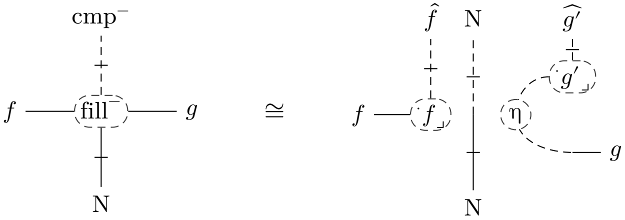
Thus we can define a *conjoinable* arrow as one with a companionable left adjoint.

Contravariant Paths

In order to fill the following “cubical horn” we need a path corresponding to the function g “backwards”:



In double categories, conjoints are dual to companions by reflecting one dimension.

Companions, conjoints and adjunctions are linked: any two determine the third; in particular:

Given an arrow adjunction $g' \dashv g$, a proarrow is a companion to g' iff it is a conjoint to g.

Thus we can define a *conjoinable* arrow as one with a companionable left adjoint.

If g is conjoinable then we can fill the square.

Path Structures on Universes

By varying which double category arrows are companionable we can model universes with various path structures.

Path Structures on Universes

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Some examples:

- if only identity arrows are companionable then the universe is path-discrete,
- if all arrows are companionable, then we have a directed path structure mirroring the function structure,
- if arrows of an adjoint equivalence are companionable, then all paths are bi-directional (because all companionable arrows then have companionable left adjoints).

References

Robert Dawson and Robert Pare. "General Associativity and General Composition for Double Categories". In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 31.1 (1993), pp. 57–79.

Marco Grandis and Robert Pare. "Limits in Double Categories". In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 40.3 (1999), pp. 162–222.

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Gregory M Kelly and Ross Street. "Review of the Elements of 2-Categories". In: *Lecture Notes in Mathematics* 420 (1974), pp. 75–103.

David Myers. "String Diagrams for Double Categories and Equipments". In: (2107). URL: https://arxiv.org/abs/1612.02762.

Michael Shulman. "Framed Bicategories and Monoidal Fibrations". In: *Theory and Application of Categories* 20.18 (2008), pp. 650–738. URL: https://arxiv.org/abs/0706.1286.

Bonus Slides

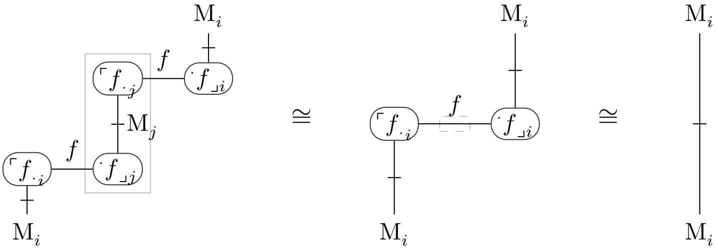
Companion Uniqueness

Theorem

If proarrows $\mathrm{M}_0, \mathrm{M}_1 : \mathrm{A} \nrightarrow \mathrm{B}$ are each companion to arrow $f : \mathrm{A} \to \mathrm{B}$ then $\mathrm{M}_0 \cong \mathrm{M}_1$.

Proof.

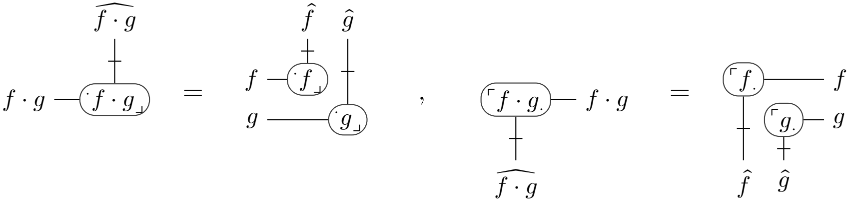
For $\{i, j\} = \{0, 1\}$, we have:



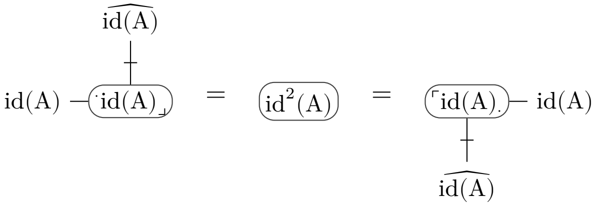
So the proarrow globes $\ulcorner f_{\cdot 1} \odot {}^{\cdot} f_{\lrcorner 0} : \mathrm{M}_0 \to \mathrm{M}_1$ and $\ulcorner f_{\cdot 0} \odot {}^{\cdot} f_{\lrcorner 1} : \mathrm{M}_1 \to \mathrm{M}_0$ form an isomorphism. □

Companion Compositionality

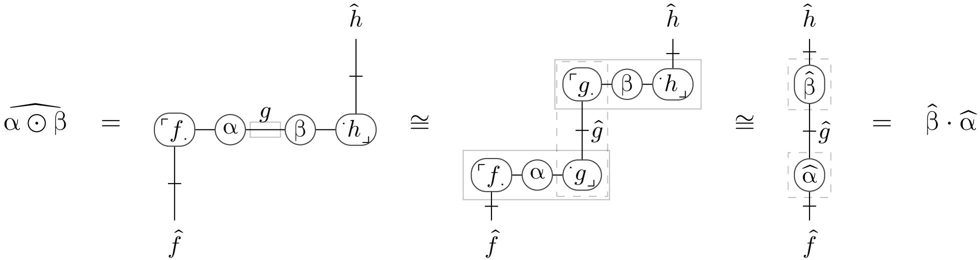
It is easily checked that the companion laws are satisfied with



and

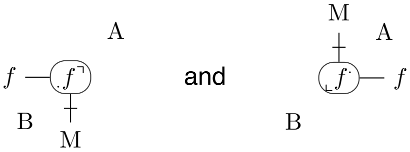


Companion Globe Compositionality

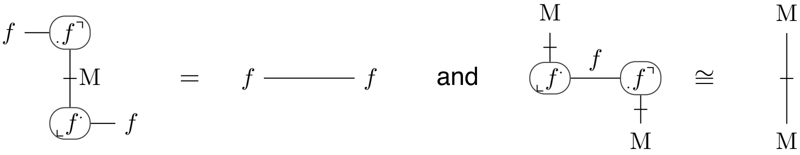


Conjoint Structure

In a double category, antiparallel arrow $f : \mathrm{A} \to \mathrm{B}$ and proarrow $\mathrm{M} : \mathrm{B} \nrightarrow \mathrm{A}$ are *conjoints* if there are *coconnection squares*



satisfying the *conjoint laws*:



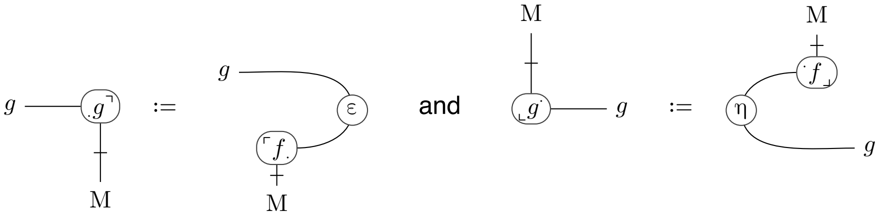
Companions, Conjoints and Adjoints

Lemma

In a double category, if arrows $f : A \to B$ and $g : B \to A$ form an arrow adjunction $f \dashv g$ then any proarrow is a companion to f just in case it is a conjoint to g.

Proof.

Suppose that proarrow $M : A \nrightarrow B$ is a companion to f. Define:



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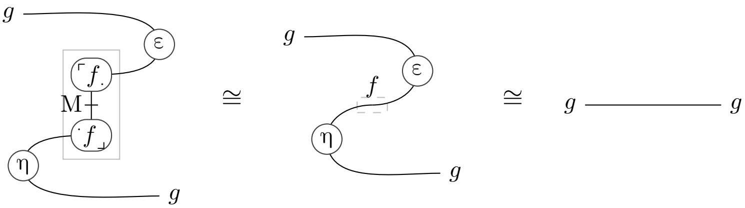
Proof.

Verify that:



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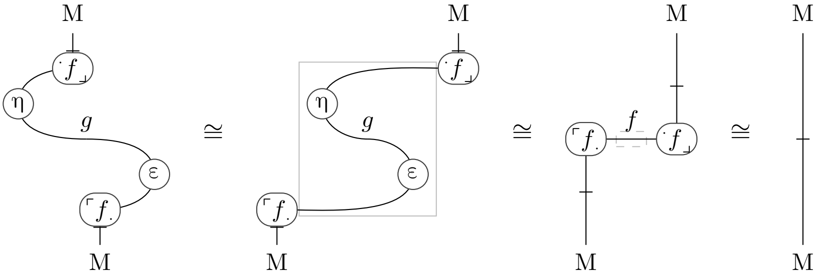
And that:



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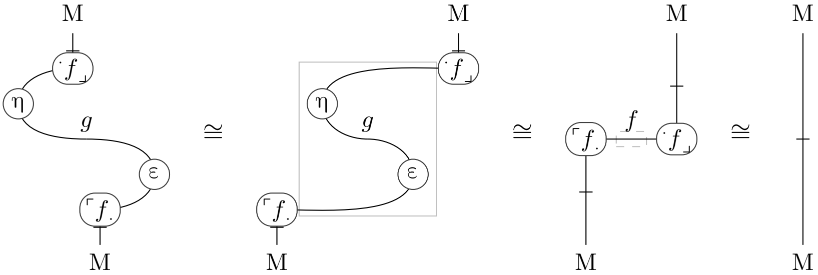

The reverse implication is dual. □