Ordered Cubes

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Context

Like *simplicial sets*, *cubical sets* provide a combinatorial model of homotopy theory.

However, there are several varieties of cubical sets to choose from.

Maps include *faces*, *degeneracies*, *diagonals*, *connections*, etc..

Relations witness properties of *geometric cubes*.

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Relations witness properties of *geometric cubes*.

Various criteria for choosing a cubical theory, including:

- homotopy theory (strict test categories),
- computational behavior (canonical forms, x-Reedy structure, distributive laws),
- model structure (judgemental vs typal equalities),
- etc.

Overview

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- contains all the familiar maps,
- has a strong equational theory,
- is a strict test category,
- is closely related to simplices.

Combinatorial Aspects

Simplicies, Order-Theoretically

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Its maps are generated by:

faces (dimension-raising maps) injective monotone functions

e.g. $d^1 = [0, 2] = \{0, 1\} \longmapsto \{0, 2\} : \Delta\left(\langle 1 \rangle \to \langle 2 \rangle\right)$

degeneracies (dimension-lowering maps) surjective monotone functions

e.g. $s^1 = [0, 1, 1] = \{0, 1, 2\} \longmapsto \{0, 1, 1\} : \Delta\left(\langle 2 \rangle \to \langle 1 \rangle\right)$

Simplicies, Monoidally

The simplex category can also be presented via the *walking monoid*, which is the category $\mathbb{M}$ with:

- one generating object, $\mathrm{V} : \mathbb{M}$
- two generating morphisms, $s : \mathbb{M}\,(\mathrm{V} \otimes \mathrm{V} \to \mathrm{V})$ and $d : \mathbb{M}\,(\mathrm{I} \to \mathrm{V})$
- relations that make (V, d, s) a monoid in $(\mathbb{M}, \otimes, \mathrm{I})$.

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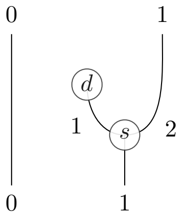
Example: composing $d^1 : \Delta\left(\langle 1 \rangle \to \langle 2 \rangle\right)$ with $s^1 : \Delta\left(\langle 2 \rangle \to \langle 1 \rangle\right)$:



Ordered (Monoidal) Cubes?

The well-studied cube categories also have order-theoretic [Jar06] and monoidal [GM03] presentations.

But in the monoidal presentation there is a “dimension mismatch”: the generating object is an *interval* rather than a *point*.

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Goal: a *vertex-based* cube category with all familiar maps and relations that is related to the simplex category by their order-theoretic presentations.

Ordered Cubes

The **standard geometric n-cube** is the convex subspace of $\mathbb{R}^n$ bounded by the 2^n vertex points $v = \underbrace{(v_0\,, \cdots, v_{n-1})}_{\text{“}v_0 \cdots v_{n-1}\text{”}}$ where $v_i \in \{0, 1\}$.

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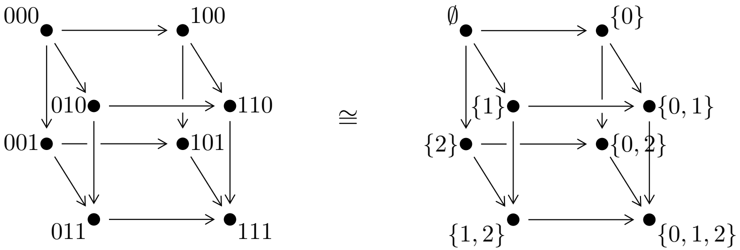
- $[n]$ is the walking product of n arrows.
- Each $[n]$ is a complete and distributive lattice.
- $[n]$ is isomorphic to the subset lattice of $\texttt{fin}(n)$ where $v_i = 1 \Leftrightarrow i \in v$:



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Among its maps are the:

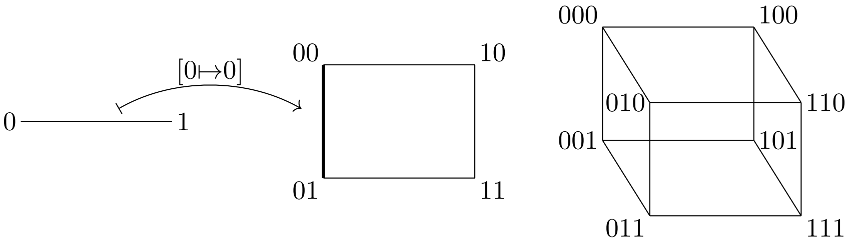
aspects (dimension-raising maps) injective monotone functions
$\square([n-1] \to [n])$

derivatives (dimension-lowering maps) surjective monotone functions
$\square([n+1] \to [n])$

Familiar Aspects

Aspects include:

Inserting coordinate $b \in \{0, 1\}$ at index i of every vertex gives a map $[i{\mapsto}b] : \square([n-1] \to [n])$ determining a **face**.



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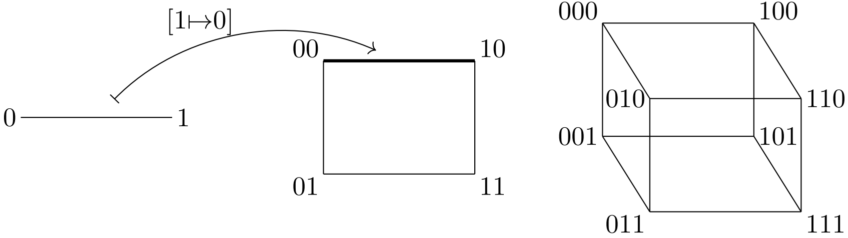

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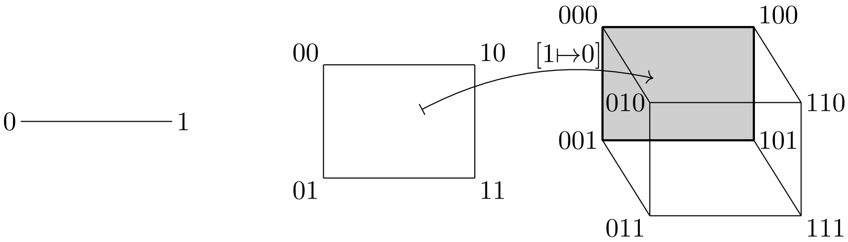

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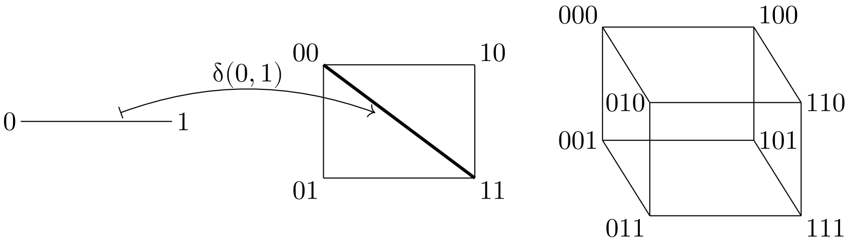

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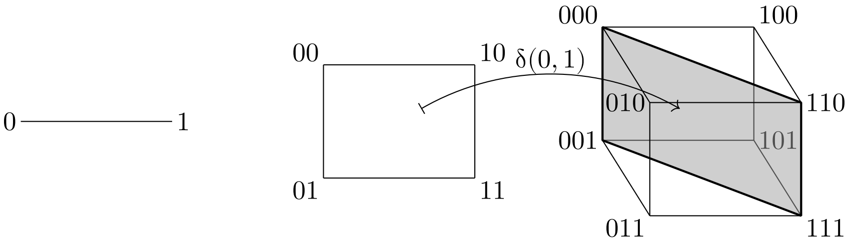

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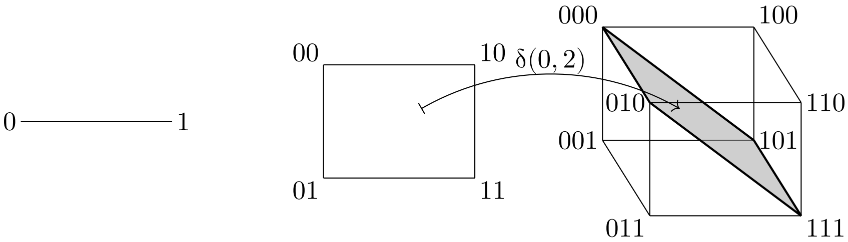

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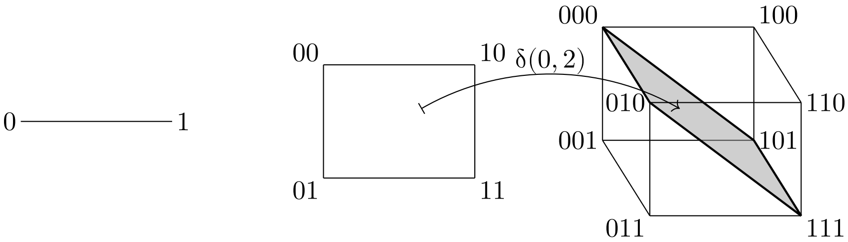

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Although drawn as polytopes, these are just order-preserving maps of vertices.

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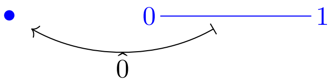

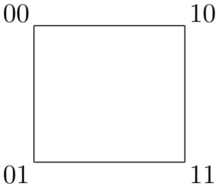

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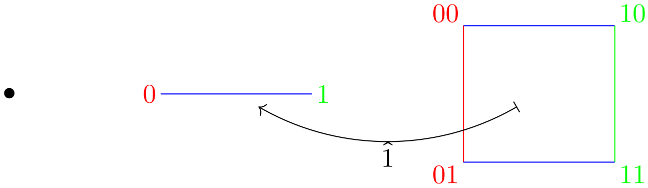

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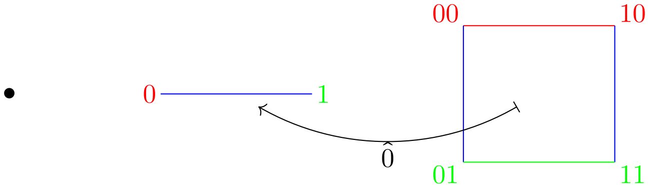

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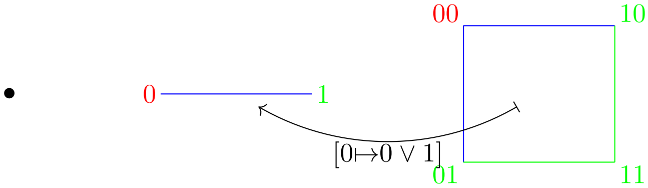

For each vertex v and $* \in \{\vee, \wedge\}$, computing the coordinate $b := v_i * v_j$, then deleting the coordinates at indices i and j, then inserting b at index k gives a map $[k{\mapsto}i * j] : \square([n+1] \to [n])$ determining a **connection.**

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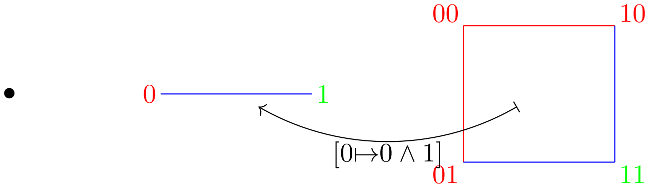

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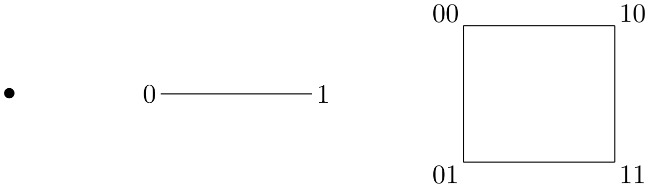

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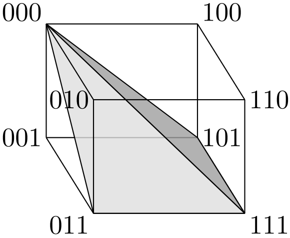
Thus $\square$ has the usual cubical maps.

Novel Maps

But there are additional maps as well,

For example, the "bent square" aspect of the cube:

$$\begin{array}{ccc} [2] & \overset{\beta}{\longrightarrow} & [3] \\ 00 & \longmapsto & 000 \\ 01 & \longmapsto & 011 \\ 10 & \longmapsto & 101 \\ 11 & \longmapsto & 111 \end{array}$$



Note: several workshop participants observed that this map is not, in fact, novel, and I am grateful to Ulrik Buchholtz for pointing out to me that the ordered cubes are equivalent to the distributive lattice cubes.

Triangulation

Since $\Delta \subseteq \textsc{Ord}$ and $\square \subseteq \textsc{Ord}$, we can consider maps in the hom $\textsc{Ord}\,(\langle m \rangle \to [n])$.

It suffices to consider the nondegenerate (i.e. injective) maps in the hom $\textsc{Ord}\,(\langle n \rangle \to [n])$.

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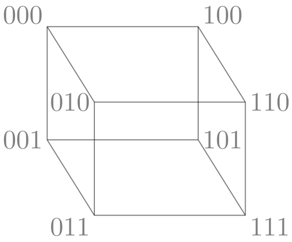

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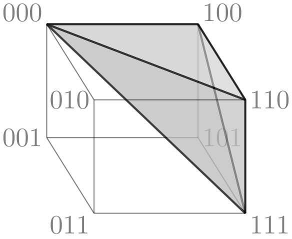

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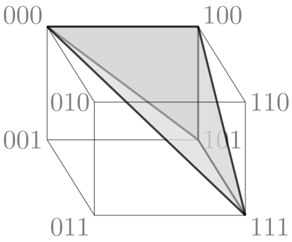

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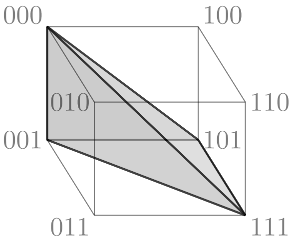

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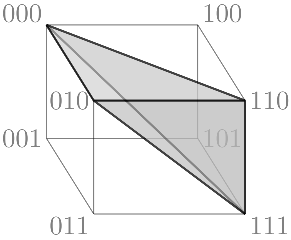

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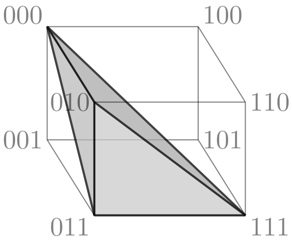

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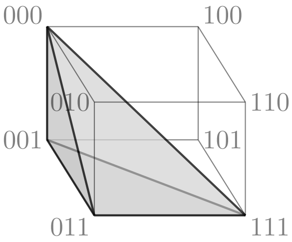

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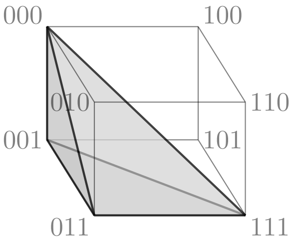

This determines a **triangulation** profunctor $t : \square \nrightarrow \Delta$ (i.e. $\Delta^\circ \times \square \to \textsc{Set}$).

Homotopical Aspects

Localization

For a category with weak equivalences $(\mathbb{C}, \mathcal{W})$ and a category $\mathbb{D}$, any functor sending weak equivalences in $\mathbb{C}$ to isos in $\mathbb{D}$

$$(\mathbb{C}, \mathcal{W}) \xrightarrow[\mathrm{F}]{} (\mathbb{D}, \mathcal{I})$$

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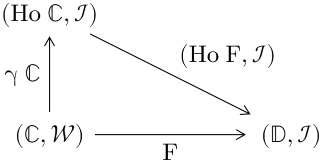

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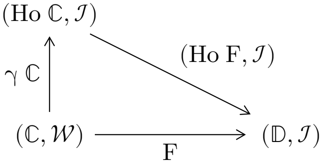
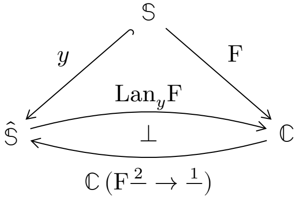


The homotopy category can be constructed by freely adding inverses to the weak equivalences.

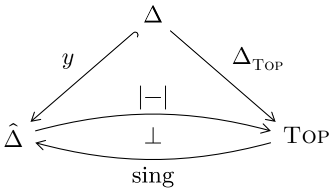
Test Categories

For small $\mathbb{S}$ and cocomplete $\mathbb{C}$, a functor $\mathrm{F} : \mathbb{S} \to \mathbb{C}$ determines an adjunction where $\mathrm{Lan}_y\mathrm{F}(\mathrm{X}) = \int^{s:\mathbb{S}} (\mathrm{X}s \otimes \mathrm{F}s)$



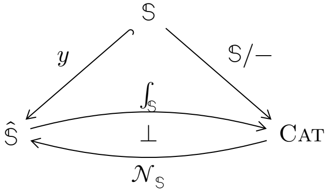
Test Categories

The standard topological simplex functor determines geometric realization and singular complex.



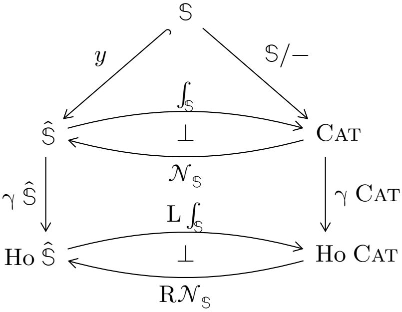
Test Categories

The slice functor determines the category of elements and nerve (where $\int_{\mathbb{S}} \mathrm{X} = y(-)/\mathrm{X}$).



Test Categories

Localization induces an adjunction on the homotopy categories.



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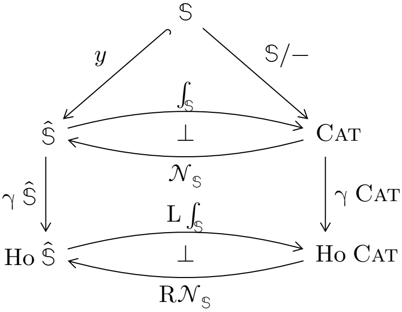

If this adjunction is an equivalence then $\mathbb{S}$ is a **weak test category.**
If this also holds true for all slices then $\mathbb{S}$ is a **test category.**
And if $\int_{\mathbb{S}} \cdot \gamma\ \mathrm{CAT}$ preserves products then $\mathbb{S}$ is a **strict test category.**

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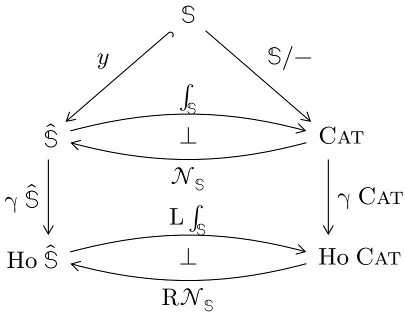

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We can do synthetic homotopy theory in the category of presheaves for any (strict) test category [Gro83].

$\square$ is a Strict Test Category

It suffices [Mal05; BM17] to observe that $\square$ has finite products:

$$1 = [0] \qquad \text{and} \qquad [m] \times [n] = [m+n]$$

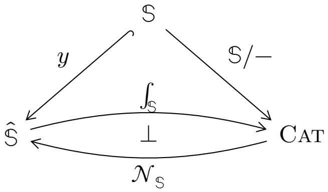
and an interval object:

$$[0{\mapsto}0], [0{\mapsto}1] : \square\,([0] \to [1])$$

whose Yoneda image is *separated* (has the unique $\hat{\square}\,(0 \to 1)$ as equalizer).

Test Functors

In the basic setup, we ask whether the slice functor induces an equivalence of homotopy categories.



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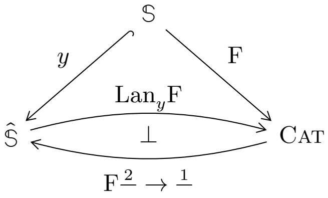

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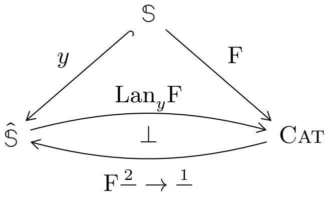

For $\mathbb{S}$ a *weak test category*, F is a **weak test functor** if:

- $\mathrm{F}(\mathrm{S})$ is *aspheric* (weakly equivalent to a point) for all $\mathrm{S} : \mathbb{S}$,
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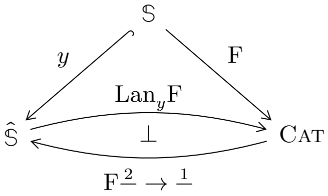

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If all slices $\partial^{-} \cdot \mathrm{F} : \mathbb{S}/\mathrm{S} \to \mathbb{S} \to \textsc{Cat}$ are weak test functors then F is a **test functor**.

$\square \hookrightarrow \textsc{Cat}$ is a Test Functor

It suffices [ZK12] to observe that $\square$ is a full subcategory of $\textsc{Cat}$ that:

- is closed under finite products,
- includes the walking interval,
- and excludes the walking nothing.

Model Structure

The category of presheaves for any test category can be equipped with a canonical *model structure* where [Cis06]:

cofibrations are the monomorphisms,

weak equivalences are the maps that become weak equivalence in $\textsc{Cat}$ under the category of elements functor.

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Fibrant objects in this model structure on $\hat{\square}$ have lots of fillings; e.g. from the "bent square" to the cube.

Implications??

Simplicial Cubes

There is a canonical functor $\square \to \hat{\Delta}$ mapping $[n] \longmapsto (\Delta^1)^{\times n}$.

Since $\hat{\Delta}$ has pointwise products (i.e. $(\mathrm{X} \times \mathrm{Y})f \cong \mathrm{X}f \times \mathrm{Y}f$), a simplex is degenerate in $\mathrm{X} \times \mathrm{Y}$ iff it is degenerate in X and Y *simultaneously*.

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Consider the nondegenerate n-simplices in $(\Delta^1)^{\times n}$.

Example: $n := 2$

$$([0,1,1]\,,[0,0,1]) \quad \text{and} \quad ([0,0,1]\,,[0,1,1])$$

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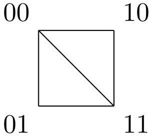

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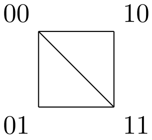
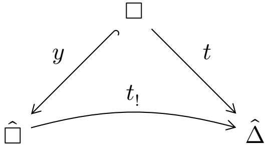


We recover the *triangulation* profunctor $t : \square \nrightarrow \Delta$.

Triangulating Cubical Sets

Since $\square$ is small and $\hat{\Delta}$ is cocomplete we can extend triangulation along Yoneda:



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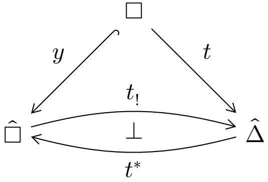

which lets us triangulate cubical sets.

This has right adjoint $t^* := \hat{\Delta}\,(t\frac{2}{} \to \frac{1}{})$ characterizing the maps from cubes into synthetic spaces presented as simplicial sets.

Summary

The ordered cubes are a shape category with good combinatorial and homotopical properties.

They may also provide an interesting foundation for a cubical type theory.

I am grateful to several workshop participants for pointing out to me related work of which I was unaware. In particular, I would like to acknowledge a recent preprint by Chris Kapulkin containing joint work done with Vladimir Voevodsky, which contains many of the results discussed here – and much more besides:
http://www.math.uwo.ca/faculty/kapulkin/papers/cubical-approach-to-straightening.pdf

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