Theoretical Question 1: The Shockley-James Paradox

In the year 1905, Albert Einstein proposed the special theory of relativity to resolve the inconsistency between Newton's mechanics and Maxwell's electromagnetism. Proper understanding of the theory led to the resolution of many apparent paradoxes. At the time, the discussion focused mostly on the propagation of electromagnetic waves.

In this question, we solve a paradox of a different type. For a fairly simple system of charges proposed by W. Shockley and R. P. James in 1967, understanding the conservation of linear momentum requires careful relativistic analysis. If a point charge is located near a magnet of changing magnetization, there's an induced electric force on the charge, but no apparent reaction on the magnet. The process may be slow enough that any electromagnetic radiation (and any momentum carried away by it) is negligible. Thus, apparently we get a cannon without recoil.

In our analysis of this system, we will demonstrate that in relativistic mechanics, a composite body may hold a non-zero mechanical momentum while remaining stationary.

Part I: Understanding the impulse on the point charge (3.3 points)

Consider a circular current loop of radius r carrying a current I_1, and a second, larger current loop of radius $R \gg r$, concentric with the first and lying in the same plane.

a. (1 pt.) A current I_2 passing through loop 2 (the larger loop) generates a magnetic flux Φ_{B1} through loop 1. Find the ratio $M_{21} = \Phi_{B1}/I_2$. It is called the mutual inductance coefficient.

b. (0.8 pts.) Given that $M_{12} = \Phi_{B2}/I_1 = M_{21}$, obtain the total induced EMF ε_2 in the larger loop as a result of a variation $\dot{I}_1 = dI_1/dt$ of the current in the smaller loop. Neglect the current in the larger loop. *Hint: the induced EMF is equal to the rate of change of the magnetic flux through the loop.*

c. (0.5 pts.) The EMF you found in part (b) is due to the tangential component of an induced electric field. Obtain an expression for the tangential electric field E at radius R as a function of the rate of change $\dot{I}_1$ of the current.

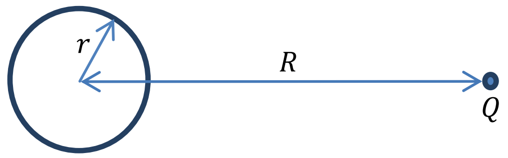

Figure 1: A circular current loop and a point charge Q.

We now remove the larger current loop, and instead put a massive point charge Q at radius R, as shown in Figure 1. It may be assumed that the charge moves very little during the relevant time periods.

d. (1 pt.) Find the total tangential impulse Δp received by the point charge as the current in the small loop changes from an initial value $I_1 = I$ to the final value $I_1 = 0$.

Part II: Understanding the recoil of the current loop (4.4 points)

We will now understand the origin of the recoil of the loop, using a loop of different geometry.

e. (1.1 pts.) Consider a hollow tube with walls made of a neutral insulating material of length l and cross section A carrying an electric current I. The current is due to charged particles of rest mass m and charge q distributed homogenously inside the tube with number density n. Assume that the charged particles are all moving along the tube with the same velocity. Find the total momentum p of the charged particles in the tube, taking Special Relativity effects into account.

f. (3.3 pts.) Consider a square current loop with side l. At a distance $R \gg l$ from the loop, there is a point charge Q; see Figure 2. The loop carries current I. We will model the current loop as a neutral tube, as in part (e). The charge carriers can move freely along the loop, colliding elastically with the walls and making elastic right turns at the corners. Neglect all interactions among the charge carriers. Assume also that all the charge carriers at a given section along the tube always move with the same velocity. Assume that the loop is heavy and that its motion can be neglected. Calculate the total linear momentum p_{hid} of the charge carriers in the loop. It is called "hidden momentum".

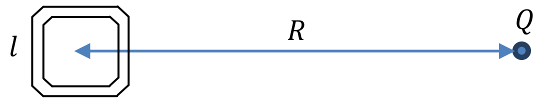

Figure 2: A square current loop and a point charge Q.

When the current stops, this linear momentum is transferred to the loop, and it gets an impulse equal to minus the impulse received by the point charge Q. This is the missing recoil that we were looking for (note that in the initial state there is also momentum in the electromagnetic field; this is important for conservation of the total momentum of the entire system).

Part III: Summarizing the results (2.3 points)

g. (0.8 pts.) Current loops are often characterized by their magnetic moment $\mu = IS$, where I is the current and S is the loop's area. Express the answer to part (d) in terms of μ, r, R and Q. Likewise, express the answer to part (f) in terms of μ, l, R and Q. Note that the electric and magnetic constants are related by:

$$\frac{4\pi k}{\mu_0} = \frac{1}{\varepsilon_0 \mu_0} = c^2$$

where c is the speed of light.

h. (1.5 pts.) In a more realistic model, the current loop is a conducting wire, and the field of the point charge Q does not penetrate into the conductor. We assume that the current is still conducted by charge carriers inside the wire. Decide whether each of the following statements is true or false, and circle the correct option in the Answer Form. **Note:** You may leave a statement undecided, but if you decide incorrectly, you will not get credit at all for part (h).

A. (0.5 pts.) The linear momentum of the current loop is zero.

B. (0.5 pts.) As the total current in the loop changes from I to zero, the charge carriers decelerate, causing induced currents in the wire's conducting material. Because of these induced currents, the point charge Q will not get a net impulse.

C. (0.5 pts.) The surface charges on the wire, induced by the presence of the external charge, will experience an electric force as the current changes from I to zero. This way, the loop will get the same impulse as found in part (f).