Theoretical Question 1
Particles and Waves

This question includes the following three parts dealing with motions of particles and waves:

Part A. Inelastic scattering of particles

Part B. Waves on a string

Part C. Waves in an expanding universe

Part A. Inelastic Scattering and Compositeness of Particles

A particle is considered ***elementary*** if it has no excitable internal degrees of freedom such as, for example, rotations and vibrations about its center of mass. Otherwise, it is ***composite***.

To determine if a particle is composite, one may set up a scattering experiment with the particle being the target and allow an *elementary* particle to scatter off it. In case that the target particle is composite, the scattering experiment may reveal important features such as *scaling*, i.e. as the forward momentum of the scattered particle increases, the scattering cross section becomes independent of the momentum.

For a scattering system consisting of an elementary particle incident on a target particle, we shall denote by Q the total translational kinetic energy *loss* of the system. Here the *translational kinetic energy* of a particle, whether elementary or composite, is defined as the kinetic energy associated with the translational motion of its center of mass. Thus we may write

$$Q = K_\mathrm{i} - K_\mathrm{f} \,,$$

where K_i and K_f are the *total* translational kinetic energies of the scattering pair before and after scattering, respectively.

In Part A, use non-relativistic classical mechanics to solve all problems. All effects due to gravity are to be neglected.

(a) As shown in Fig. 1, an *elementary* particle of mass m moves along the x axis with x-component of momentum $p_1 > 0$. After being scattered by a *stationary* target of mass M, its momentum becomes $\vec{p}_2$.

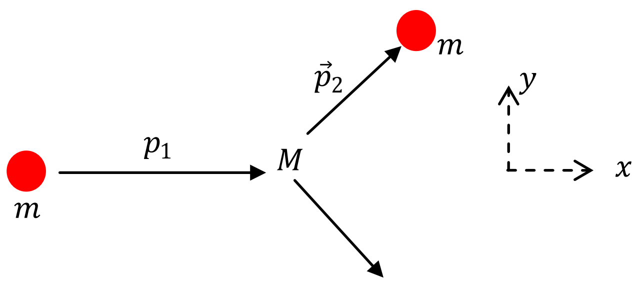

Fig. 1

From data on $\vec{p}_2$, one can determine if the target particle is elementary or composite. We shall assume that $\vec{p}_2$ lies in the x-y plane and that the x- and y-components of $\vec{p}_2$ are given, respectively, by p_{2x} and p_{2y}.

(i) Find an expression for Q in terms of m, M, p_1, p_{2x}, and p_{2y}. [0.2 point]

(ii) If the target particle is elementary, the momenta p_1, p_{2x}, and p_{2y} are related in a particular way by a condition.

For given p_1, plot the condition as a curve in the p_{2x}-p_{2y} plane. Specify the value of p_{2x} for each intercept of the curve with the p_{2x}-axis. In the same plot, locate regions of points of $\vec{p}_2$ corresponding to $Q < 0$, $Q = 0$, $Q > 0$, and label each of them as such.

[0.7 point]

For a stationary composite target in its ground state before scattering, which region(s) of Q contains those points of $\vec{p}_2$ allowed? [0.2 point]

(b) Now, consider a composite target consisting of two elementary particles each with mass $\frac{1}{2}M$. They are connected by a spring of negligible mass. See Fig. 2. The spring has a force constant k and does not bend sideways. Initially, the target is *stationary* with its center of mass at the origin O, and the spring, inclined at an angle θ to the x-axis, is at its natural length d_0. For simplicity, we assume that only vibrational and rotational motions can be excited in the target as a result of scattering.

The incident elementary particle of mass m moves in the x-direction both before and after scattering with its momenta given, respectively, by p_1 and p_2. Note that p_2 is negative if the particle recoils and moves backward. A scattering occurs only if the incident particle hits one of the target particles and $p_2 \neq p_1$. We assume all three particles move in the same plane before and after scattering.

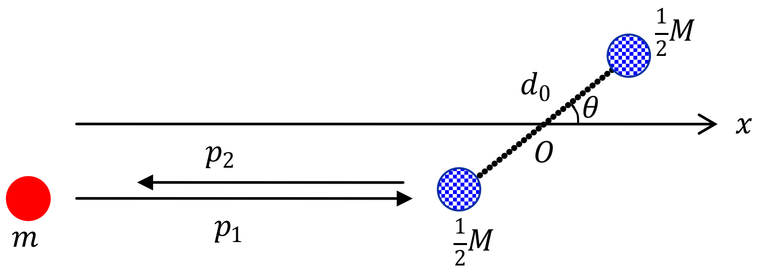

Fig. 2

(i) If the maximum length of the spring after scattering is d_m, find an equation which relates the ratio $x = (d_\mathrm{m} - d_0)/d_0$ to the quantities $Q, \theta, d_0, m, k, M, p_1$ and p_2.

[0.7 point]

(ii) Let $\alpha \equiv \sin^2\theta$. When the angle of orientation θ of the target is allowed to vary, the scattering cross section σ gives the effective target area, in a plane normal to the direction of incidence, which allows certain outcomes to occur as a result of scattering. It is known that for all outcomes which lead to the same value of p_2, the value of α must span an interval $(\alpha_{\min}, \alpha_{\max})$ and we may choose the unit of cross section so that σ is simply given by the numerical range $(\alpha_{\max} - \alpha_{\min})$ of the interval. Note that $\alpha_{\min}$, $\alpha_{\max}$, and, consequently, σ are dependent on p_2. Let p_c be the threshold value of p_2 at which σ starts to become independent of p_2.

In the limit of large k, give an estimate of p_c. Express your answer in terms of m, M, and p_1. [1.1 points]

Assume M=3m and in the limit of large k, plot σ as a function of p_2 for a given p_1. In the plot, specify the range of σ and p_2. [1.1 points]

Part B. Waves on a String

Consider an elastic string stretched between two fixed ends A and B, as shown in Fig. 3. The linear mass density of the string is μ. The speed of propagation for transverse waves in the string is c. Let the length $\overline{AB}$ be L. The string is plucked sideways and held in a triangular form with a maximum height $h \ll L$ at its middle point. At time $t = 0$, the plucked string is released from rest. All effects due to gravity may be neglected.

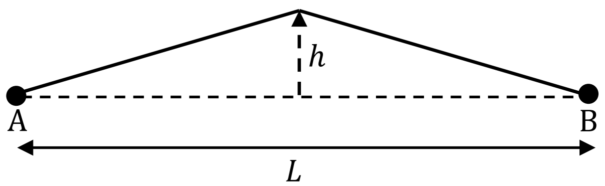

Fig. 3

(c) Find the period of vibration T for the string. [0.5 point]

Plot the shape of the string at $t = T/8$. In the plot, specify lengths and angles which serve to define the shape of the string. [1.7 points]

(d) Find the total mechanical energy of the vibrating string in terms of μ, c, h, and L. [0.8 point]

Part C. An Expanding Universe

Photons in the universe play an important role in delivering information across the cosmos. However, the fact that the universe is expanding must be taken into account when one tries to extract information from these photons. To this end, we normally express length and distance using a universal scale factor $a(t)$ which depends on time t. Thus the distance $L(t)$ between two stars stationary in their respective local frames is proportional to $a(t)$:

$$L(t) = ka(t), \qquad (1)$$

where k is a constant and $a(t)$ accounts for the expansion of the universe. We use a dot above a symbol of a variable to denote its time derivative, i.e. $\dot{a}(t) = da(t)/dt$, and let $v(t) \equiv \dot{L}(t)$. Taking time derivatives of both sides of Eq. (1), one obtains the Hubble law:

$$v(t) = H(t)L(t), \qquad (2)$$

where $H(t) = \dot{a}(t)/a(t)$ is the *Hubble parameter* at time t. At the current time t_0, we have

$$H(t_0) = 72 \text{ km s}^{-1} \text{ Mpc}^{-1},$$

where 1 Mpc $= 3.0857 \times 10^{19}$ km $= 3.2616 \times 10^{6}$ light-year.

Assume the universe to be infinitely large and expanding in such a way that

$$a(t) \propto \exp(bt),$$

where b is a constant. In such a universe, the Hubble parameter is a constant equal to $H(t_0)$. Moreover, it can be shown that the wavelength λ of photons travelling in the universe will be stretched in proportion to the expansion of the universe, i.e.

$$\lambda(t) \propto a(t).$$

Now suppose that photons making up a Lyman-alpha emission line were emitted at t_e by a star that was stationary in its local frame and that we as observers are stationary in our local frame. When these photons were emitted, their wavelength was $\lambda(t_e) = 121.5$ nm. But when they reach us now at t_0, their wavelength is red-shifted to 145.8 nm.

(e) As these photons traveled, the universe kept expanding so that the star kept receding from us. Given that the speed of light in vacuum c has never changed, what *was* the distance $L(t_e)$ of the star from us when these photons were emitted at t_e? Express the answer in units of Mpc. [2.2 points]

(f) What is the receding velocity $v(t_0)$ of the star with respect to us *now* at t_0? Express the answer in units of the speed of light in vacuum c. [0.8 point]

Appendix

The following formula may be used when needed:

$$\int_a^b e^{\beta x}\, dx = \frac{1}{\beta}\left(e^{\beta b} - e^{\beta a}\right).$$