Problems of the 20th International Physics Olympiad [1]
(Warsaw, 1989)

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Abstract

The article contains problems given at the 20th International Physics Olympiad (1989) and their solutions. The 20th IPhO was the third IPhO organized in Warsaw, Poland.

Logo

The emblem of the XX International Physics Olympiad contains a picture that is a historical record of the first hypernuclear event observed and interpreted in Warsaw by M. Danysz and J. Pniewski[3]. The collision of a high-energy particle with a heavy nucleus was registered in nuclear emulsion. Tracks of the secondary particles emitted in the event, seen in the picture (upper star), consist of tracks due to fast pions ("thin tracks") and to much slower fragments of the target nucleus ("black tracks"). The "black track" connecting the upper star (greater) with the lower star (smaller) in the figure is due to a hypernuclear fragment, in this case due to a part of the primary nucleus containing an unstable hyperon Λ instead of a nucleon. Hyperfragments (hypernuclei) are a new kind of matter in which the nuclei contain not only protons and neutrons but also some other heavy particles.

In the event observed above the hyperon Λ, bound with nucleon, decays like a free particle through a week (slow) process only. This fact strongly suggested the existence of a new quantum number that could explain suppression of the decay, even in presence of nucleons. Indeed, this was one of the observations that, 30 months later, led to the concept of strangeness.

Introduction

Theoretical problems (including solutions and marking schemes) were prepared especially for the 20th IPhO by Waldemar Gorzkowski. The experimental problem (including the solution and marking scheme) was prepared especially for this Olympiad by Andrzej Kotlicki. The problems were refereed independently (and many times) by at least two persons

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[3] M. Danysz and J. Pniewski, *Bull. Acad. Polon. Sci.*, **3(1)** 42 (1952) and *Phil. Mag.*, **44**, 348 (1953). Later the same physicists, Danysz and Pniewski, discovered the first case of a nucleus with two hyperons (double hyperfragment).

after any change was made in the text to avoid unexpected difficulties at the competition. This work was done by:

First Problem:
Andrzej Szadkowski, Andrzej Szymacha, Włodzimierz Ungier

Second Problem:
Andrzej Szadkowski, Andrzej Szymacha, Włodzimierz Ungier, Stanisław Woronowicz

Third Problem:
Andrzej Rajca, Andrzej Szymacha, Włodzimierz Ungier

Experimental Problem:
Krzysztof Korona, Anna Lipniacka, Jerzy Łusakowski, Bruno Sikora

Several English versions of the texts of the problems were given to the English-speaking students. As far as I know it happened for the first time (at present it is typical). The original English version was accepted (as a version for the students) by the leaders of the Australian delegation only. The other English-speaking delegations translated the English originals into English used in their countries. The net result was that there were at least four English versions. Of course, physics contained in them was exactly the same, while wording and spelling were somewhat different (the difference, however, were not too great).

This article is based on the materials quoted at the end of the article and on personal notes of the author.

THEORETICAL PROBLEMS

Problem 1

Consider two liquids A and B insoluble in each other. The pressures p_i (i = A or B) of their saturated vapors obey, to a good approximation, the formula:

$$\ln(p_i / p_o) = \frac{\alpha_i}{T} + \beta_i; \quad i = \text{A or B,}$$

where p_o denotes the normal atmospheric pressure, T – the absolute temperature of the vapor, and α_i and β_i (i = A or B) – certain constants depending on the liquid. (The symbol ln denotes the natural logarithm, i.e. logarithm with base e = 2.7182818…)

The values of the ratio p_i/p_0 for the liquids A and B at the temperature 40°C and 90°C are given in Tab. 1.1.

Table 1.1

t [°C]	p_i/p_0	
	$i = \text{A}$	$i = \text{B}$
40	0.284	0.07278
90	1.476	0.6918

The errors of these values are negligible.

A. Determine the boiling temperatures of the liquids A and B under the pressure p_0.

B. The liquids A and B were poured into a vessel in which the layers shown in Fig. 1.1 were formed. The surface of the liquid B has been covered with a thin layer of a non-volatile liquid C, which is insoluble in the liquids A and B and vice versa, thereby preventing any free evaporation from the upper surface of the liquid B, The ratio of molecular masses of the liquids A and B (in the gaseous phase) is:
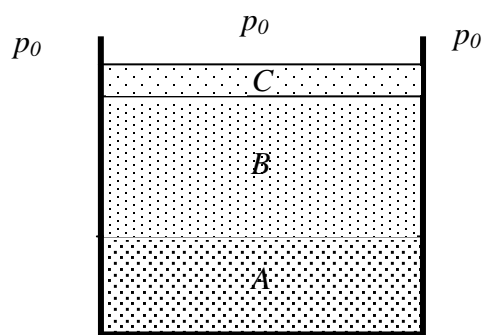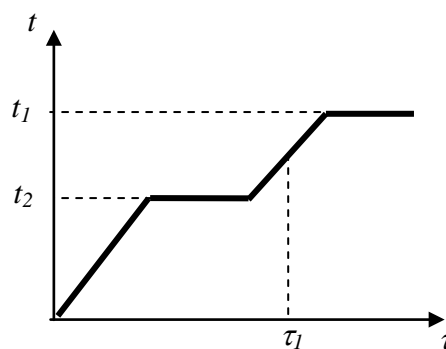
$$\gamma = \mu_A / \mu_B = 8.$$

Fig. 1.1   Fig. 1.2

The masses of the liquids A and B were initially the same, each equal to $m = 100$g. The heights of the layers of the liquids in the vessel and the densities of the liquids are small enough to make the assumption that the pressure in any point in the vessel is practically equal to the normal atmospheric pressure p_0.

The system of liquids in the vessel is slowly, but continuously and uniformly, heated. It was established that the temperature t of the liquids changed with time τ as shown schematically in the Fig. 1.2.

Determine the temperatures t_1 and t_2 corresponding to the horizontal parts of the diagram and the masses of the liquids A and B at the time τ_1. The temperatures should be rounded to the nearest degree (in °C) and the masses of the liquids should be determined to one-tenth of gram.

REMARK: Assume that the vapors of the liquids, to a good approximation,

(1) obey the Dalton law stating that the pressure of a mixture of gases is equal to the sum of the partial pressures of the gases forming the mixture and
(2) can be treated as perfect gases up to the pressures corresponding to the saturated vapors.

Solution

PART A

The liquid boils when the pressure of its saturated vapor is equal to the external pressure. Thus, in order to find the boiling temperature of the liquid i (i - A or B), one should determine such a temperature T_{bi} (or t_{bi}) for which $p_i/p_0 = 1$.

Then $\ln(p_i / p_0) = 0$, and we have:

$$T_{bi} = -\frac{\alpha_i}{\beta_i}.$$

The coefficients α_i and β_i are not given explicitly. However, they can be calculated from the formula given in the text of the problem. For this purpose one should make use of the numerical data given in the Tab. 1.1.

For the liquid A, we have:

$$\ln 0.284 = \frac{\alpha_A}{(40 + 273.15)\text{K}} + \beta_A,$$

$$\ln 1.476 = \frac{\alpha_A}{(90 + 273.15)\text{K}} + \beta_A.$$

After subtraction of these equations, we get:

$$\ln 0.284 - \ln 1.476 = \alpha_A \left(\frac{1}{40 + 273.15} - \frac{1}{90 + 273.15} \right) \text{K}^{-1}.$$

$$\alpha_A = \frac{\ln \dfrac{0.284}{1.476}}{\dfrac{1}{40 + 273.15} - \dfrac{1}{90 + 273.15}} \text{K} \approx -3748.49\text{K}.$$

Hence,

$$\beta_A = \ln 0.284 - \frac{\alpha_A}{(40 + 273.15)\text{K}} \approx 10.711.$$

Thus, the boiling temperature of the liquid A is equal to

$$T_{bA} = 3748.49\text{K}/10.711 \approx 349.95 \text{ K.}$$

In the Celsius scale the boiling temperature of the liquid A is

$$t_{bA} = (349.95 - 273.15)°\text{C} = 76.80°\text{C} \approx 77°\text{C.}$$

For the liquid B, in the same way, we obtain:

$$\alpha_B \approx -5121.64 \text{ K,}$$
$$\beta_B \approx 13.735,$$
$$T_{bB} \approx 372\text{-}89 \text{ K,}$$
$$t_{bB} \approx 99.74°\text{C} \approx 100°\text{C.}$$

PART B

As the liquids are in thermal contact with each other, their temperatures increase in time in the same way.

At the beginning of the heating, what corresponds to the left sloped part of the diagram, no evaporation can occur. The free evaporation from the upper surface of the liquid B cannot occur - it is impossible due to the layer of the non-volatile liquid C. The evaporation from the inside of the system is considered below.

Let us consider a bubble formed in the liquid A or in the liquid B or on the surface that separates these liquids. Such a bubble can be formed due to fluctuations or for many other reasons, which will not be analyzed here.

The bubble can get out of the system only when the pressure inside it equals to the external pressure p_0 (or when it is a little bit higher than p_0). Otherwise, the bubble will collapse.

The pressure inside the bubble formed in the volume of the liquid A or in the volume of the liquid B equals to the pressure of the saturated vapor of the liquid A or B, respectively. However, the pressure inside the bubble formed on the surface separating the liquids A and B is equal to the sum of the pressures of the saturated vapors of both these liquids, as then the bubble is in a contact with the liquids A and B at the same time. In the case considered the pressure inside the bubble is greater than the pressures of the saturated vapors of each of the liquids A and B (at the same temperature).

Therefore, when the system is heated, the pressure p_0 is reached first in the bubbles that were formed on the surface separating the liquids. Thus, the temperature t_1 corresponds to a kind of common boiling of both liquids that occurs in the region of their direct contact. The temperature t_1 is for sure lower than the boiling temperatures of the liquids A and B as then the pressures of the saturated vapors of the liquids A and B are less then p_0 (their sum equals to p_0 and each of them is greater than zero).

In order to determine the value of t_1 with required accuracy, we can calculate the values of the sum of the saturated vapors of the liquids A and B for several values of the temperature t and look when one gets the value p_0.

From the formula given in the text of the problem, we have:

$$\frac{p_A}{p_0} = e^{\frac{\alpha_A}{T} + \beta_A}, \tag{1}$$

$$\frac{p_B}{p_0} = e^{\frac{\alpha_B}{T} + \beta_B}. \tag{2}$$

$p_A + p_B$ equals to p_0 if

$$\frac{p_A}{p_0} + \frac{p_B}{p_0} = 1.$$

Thus, we have to calculate the values of the following function:

$$y(x) = e^{\frac{\alpha_A}{t+t_0} + \beta_A} + e^{\frac{\alpha_B}{t+t_0} + \beta_B},$$

(where $t_0 = 273.15\,°C$) and to determine the temperature $t = t_1$, at which $y(t)$ equals to 1. When calculating the values of the function $y(t)$ we can divide the intervals of the temperatures t by 2 (approximately) and look whether the results are greater or less than 1.

We have:

Table 1.2

t	$y(t)$
40°C	< 1 (see Tab. 1.1)
77°C	> 1 (as t_1 is less than t_{bA})
59°C	0.749 < 1
70°C	1.113 > 1
66°C	0.966 < 1
67°C	1.001 > 1
66.5°C	0.983 < 1

Therefore, $t_1 \approx 67°$ C (with required accuracy).

Now we calculate the pressures of the saturated vapors of the liquids A and B at the temperature $t_1 \approx 67°C$, i.e. the pressures of the saturated vapors of the liquids A and B in each bubble formed on the surface separating the liquids. From the equations (1) and (2), we get:

$$p_A \approx 0.734 \, p_0,$$
$$p_B \approx 0.267 \, p_0,$$
$$(p_A + p_B = 1.001 p_0 \approx p_0).$$

These pressures depend only on the temperature and, therefore, they remain constant during the motion of the bubbles through the liquid B. The volume of the bubbles during this motion also cannot be changed without violation of the relation $p_A + p_B = p_0$. It follows from the above remarks that the mass ratio of the saturated vapors of the liquids A and B in each bubble is the same. This conclusion remains valid as long as both liquids are in the system. After total evaporation of one of the liquids the temperature of the system will increase again (second sloped part of the diagram). Then, however, the mass of the system remains constant until the temperature reaches the value t_2 at which the boiling of the liquid (remained in the vessel) starts. Therefore, the temperature t_2 (the higher horizontal part of the diagram) corresponds to the boiling of the liquid remained in the vessel.

The mass ratio m_A / m_B of the saturated vapors of the liquids A and B in each bubble leaving the system at the temperature t_1 is equal to the ratio of the densities of these vapors ρ_A / ρ_B. According to the assumption 2, stating that the vapors can be treated as ideal gases, the last ratio equals to the ratio of the products of the pressures of the saturated vapors by the molecular masses:

$$\frac{m_A}{m_B} = \frac{\rho_A}{\rho_B} = \frac{p_A \mu_A}{p_B \mu_B} = \frac{p_A}{p_B} \mu .$$

Thus,

$$\frac{m_A}{m_B} \approx 22.0 .$$

We see that the liquid A evaporates 22 times faster than the liquid B. The evaporation of 100 g of the liquid A during the "surface boiling" at the temperature t_1 is associated with the

evaporation of 100 g / 22 ≈ 4.5 g of the liquid B. Thus, at the time τ_1 the vessel contains 95.5 g of the liquid B (and no liquid A). The temperature t_2 is equal to the boiling temperature of the liquid B: $t_2 = 100°C$.

Marking Scheme

1.	physical condition for boiling	1 point
2.	boiling temperature of the liquid A (numerical value)	1 point
3.	boiling temperature of the liquid B (numerical value)	1 point
4.	analysis of the phenomena at the temperature t_1	3 points
5.	numerical value of t_1	1 point
6.	numerical value of the mass ratio of the saturated vapors in the bubble	1 point
7.	masses of the liquids at the time τ_1	1 point
8.	determination of the temperature t_2	1 point

REMARK: As the sum of the logarithms is not equal to the logarithm of the sum, the formula given in the text of the problem should not be applied to the mixture of the saturated vapors in the bubbles formed on the surface separating the liquids. However, the numerical data have been chosen in such a way that even such incorrect solution of the problem gives the correct value of the temperature t_1 (within required accuracy). The purpose of that was to allow the pupils to solve the part B of the problem even if they determined the temperature t_1 in a wrong way. Of course, one cannot receive any points for an incorrect determination of the temperature t_1 even if its numerical value is correct.

Typical mistakes in the pupils' solutions

Nobody has received the maximum possible number of points for this problem, although several solutions came close. Only two participants tried to analyze proportion of pressures of the vapors during the upward movement of the bubble trough the liquid B. Part of the students confused Celsius degrees with Kelvins. Many participants did not take into account the boiling on the surface separating the liquids A and B, although this effect was the essence of the problem. Part of the students, who did notice this effect, assumed a priori that the liquid with lower boiling temperature "must" be the first to evaporate. In general, this need not be true: if γ were, for example, 1/8 instead 8, then liquid A rather than B would remain in the vessel. As regards the boiling temperatures, practically nobody had any essential difficulties.

Problem 2

Three non-collinear points P_1, P_2 and P_3, with known masses m_1, m_2 and m_3, interact with one another through their mutual gravitational forces only; they are isolated in free space and do not interact with any other bodies. Let σ denote the axis going through the center-of-mass of the three masses, and perpendicular to the triangle $P_1P_2P_3$. What conditions should the angular velocities ω of the system (around the axis σ) and the distances:

$$P_1P_2 = a_{12}, \qquad P_2P_3 = a_{23}, \qquad P_1P_3 = a_{13},$$

fulfill to allow the shape and size of the triangle $P_1P_2P_3$ unchanged during the motion of the system, i.e. under what conditions does the system rotate around the axis σ as a rigid body?

Solution

As the system is isolated, its total energy, i.e. the sum of the kinetic and potential energies, is conserved. The total potential energy of the points P_1, P_2 and P_3 with the masses m_1, m_2 and m_3 in the inertial system (i.e. when there are no inertial forces) is equal to the sum of the gravitational potential energies of all the pairs of points (P_1,P_2), (P_2,P_3) and (P_1,P_3). It depends only on the distances a_{12}, a_{23} and a_{23} which are constant in time. Thus, the total potential energy of the system is constant. As a consequence the kinetic energy of the system is constant too. The moment of inertia of the system with respect to the axis σ depends only on the distances from the points P_1, P_2 and P_3 to the axis σ which, for fixed a_{12}, a_{23} and a_{23} do not depend on time. This means that the moment of inertia I is constant. Therefore, the angular velocity of the system must also be constant:

$$\omega = \text{const.} \tag{1}$$

This is the first condition we had to find. The other conditions will be determined by using three methods described below. However, prior to performing calculations, it is desirable to specify a convenient coordinates system in which the calculations are expected to be simple.

Let the positions of the points P_1, P_2 and P_3 with the masses m_1, m_2 and m_3 be given by the vectors $\mathbf{r_1}$, $\mathbf{r_2}$ and $\mathbf{r_3}$. For simplicity we assume that the origin of the coordinate system is localized at the center of mass of the points P_1, P_2 and P_3 with the masses m_1, m_2 and m_3 and that all the vectors $\mathbf{r_1}$, $\mathbf{r_2}$ and $\mathbf{r_3}$ are in the same coordinate plane, e.g. in the plane (x,y). Then the axis σ is the axis z.

In this coordinate system, according to the definition of the center of mass, we have:

$$m_1\mathbf{r_1} + m_2\mathbf{r_2} + m_3\mathbf{r_2} = 0 \tag{2}$$

Now we will find the second condition by using several methods.

FIRST METHOD

Consider the point P_1 with the mass m_1. The points P_2 and P_3 act on it with the forces:

$$\mathbf{F}_{21} = G\frac{m_1 m_2}{a_{12}^3}(\mathbf{r_2} - \mathbf{r_1}), \tag{3}$$

$$\mathbf{F}_{31} = G\frac{m_1 m_3}{a_{13}^3}(\mathbf{r_3} - \mathbf{r_1}). \tag{4}$$

where G denotes the gravitational constant.

In the inertial frame the sum of these forces is the centripetal force

$$\mathbf{F}_{r1} = -m_1\omega^2\mathbf{r_1},$$

which causes the movement of the point P_1 along a circle with the angular velocity ω. (The moment of this force with respect to the axis σ is equal to zero.) Thus, we have:

$$\mathbf{F}_{21} + \mathbf{F}_{31} = \mathbf{F}_{r1}. \tag{5}$$

In the non-inertial frame, rotating around the axis σ with the angular velocity ω, the sum of the forces (3), (4) and the centrifugal force

$$\mathbf{F'}_{r1} = m_1 \omega^2 \mathbf{r}_1$$

should be equal to zero:

$$\mathbf{F}_{21} + \mathbf{F}_{31} + \mathbf{F'}_{r1} = 0. \tag{6}$$

(The moment of this sum with respect to any axis equals to zero.)

The conditions (5) and (6) are equivalent. They give the same vector equality:

$$G\frac{m_1 m_2}{a_{12}^3}(\mathbf{r}_2 - \mathbf{r}_1) + G\frac{m_1 m_3}{a_{13}^3}(\mathbf{r}_3 - \mathbf{r}_1) + m_1 \omega^2 \mathbf{r}_1 = 0, \tag{7'}$$

$$G\frac{m_1}{a_{12}^3}m_2\mathbf{r}_2 + G\frac{m_1}{a_{13}^3}m_3\mathbf{r}_3 + m_1\mathbf{r}_1\left(\omega^2 - \frac{Gm_2}{a_{12}^3} - \frac{Gm_3}{a_{13}^3}\right) = 0 \tag{7''}$$

From the formula (2), we get:

$$m_2\mathbf{r}_2 = -m_1\mathbf{r}_1 - m_3\mathbf{r}_3 \tag{8}$$

Using this relation, we write the formula (7) in the following form:

$$G\frac{m_1}{a_{12}^3}(-m_1\mathbf{r}_1 - m_3\mathbf{r}_3) + G\frac{m_1}{a_{13}^3}m_3\mathbf{r}_3 + m_1\mathbf{r}_1\left(\omega^2 - \frac{Gm_2}{a_{12}^3} - \frac{Gm_3}{a_{13}^3}\right) = 0 ,$$

i.e.

$$\mathbf{r}_1 m_1\left(\omega^2 - \frac{Gm_2}{a_{12}^3} - \frac{Gm_3}{a_{13}^3} - \frac{Gm_1}{a_{12}^3}\right) + \mathbf{r}_3\left(\frac{1}{a_{13}^3} - \frac{1}{a_{12}^3}\right)Gm_1m_3 = 0.$$

The vectors $\mathbf{r}_1$ and $\mathbf{r}_3$ are non-collinear. Therefore, the coefficients in the last formula must be equal to zero:

$$\left(\frac{1}{a_{13}^3} - \frac{1}{a_{12}^3}\right)Gm_1m_3 = 0,$$

$$m_1\left(\omega^2 - \frac{Gm_2}{a_{12}^3} - \frac{Gm_3}{a_{13}^3} - \frac{Gm_1}{a_{12}^3}\right) = 0.$$

The first equality leads to:

$$\frac{1}{a_{13}^3} = \frac{1}{a_{12}^3}$$

and hence,

$$a_{13} = a_{12}.$$

Let $a_{13} = a_{12} = a$. Then the second equality gives:

$$\omega^2 a^3 = GM \tag{9}$$

where

$$M = m_1 + m_2 + m_3 \tag{10}$$

denotes the total mass of the system.

In the same way, for the points P_2 and P_3, one gets the relations:

a) the point P_2:

$$a_{23} = a_{12}; \quad \omega^2 a^3 = GM$$

b) the point P_3:

$$a_{13} = a_{23}; \quad \omega^2 a^3 = GM$$

Summarizing, the system can rotate as a rigid body if all the distances between the masses are equal:

$$a_{12} = a_{23} = a_{13} = a, \tag{11}$$

the angular velocity ω is constant and the relation (9) holds.

SECOND METHOD

At the beginning we find the moment of inertia I of the system with respect to the axis σ. Using the relation (2), we can write:

$$0 = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3)^2 = m_1^2\mathbf{r}_1^2 + m_2^2\mathbf{r}_2^2 + m_3^2\mathbf{r}_3^2 + 2m_1m_2\mathbf{r}_1\mathbf{r}_2 + 2m_1m_3\mathbf{r}_1\mathbf{r}_3 + 2m_3m_2\mathbf{r}_3\mathbf{r}_2.$$

Of course,

$$\mathbf{r}_i^2 = r_i^2 \qquad i = 1, 2, 3$$

The quantities $2\mathbf{r}_i\mathbf{r}_j$ $(i, j = 1, 2, 3)$ can be determined from the following obvious relation:

$$a_{ij}^2 = \left|\mathbf{r}_i - \mathbf{r}_j\right|^2 = (\mathbf{r}_i - \mathbf{r}_j)^2 = \mathbf{r}_i^2 + \mathbf{r}_j^2 - 2\mathbf{r}_i\mathbf{r}_j = r_i^2 + r_j^2 - 2\mathbf{r}_i\mathbf{r}_j.$$

We get:

$$2\mathbf{r}_i\mathbf{r}_j = r_i^2 + r_j^2 - a_{ij}^2.$$

With help of this relation, after simple transformations, we obtain:

$$0 = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3)^2 = (m_1 + m_2 + m_3)(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2) - \sum_{i<j} m_i m_j a_{ij}^2.$$

The moment of inertia I of the system with respect to the axis σ, according to the definition of this quantity, is equal to

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2.$$

The last two formulae lead to the following expression:

$$I = \frac{1}{M} \sum_{i<j} m_i m_j a_{ij}^2$$

where M (the total mass of the system) is defined by the formula (10).

In the non-inertial frame, rotating around the axis σ with the angular velocity ω, the total potential energy V_{tot} is the sum of the gravitational potential energies

$$V_{ij} = -G\frac{m_i m_j}{a_{ij}}; \qquad\qquad i, j = 1, 2, 3; i < j$$

of all the masses and the potential energies

$$V_i = -\tfrac{1}{2}\omega^2 m_i r_i^2 ; \qquad\qquad i = 1,\, 2,\, 3$$

of the masses m_i $(i = 1,\, 2,\, 3)$ in the field of the centrifugal force:

$$V_{tot} = G\sum_{i<j}\frac{m_i m_j}{a_{ij}} - \tfrac{1}{2}\omega^2\sum_{i=1}^{3} m_i r_i^2 = G\sum_{i<j}\frac{m_i m_j}{a_{ij}} - \tfrac{1}{2}\omega^2 I = G\sum_{i<j}\frac{m_i m_j}{a_{ij}} - \tfrac{1}{2}\omega^2\frac{1}{M}\sum_{i<j} m_i m_j a_{ij}^2 =$$

$$= -\sum_{i<j} m_i m_j\left(\frac{\omega^2}{2M}a_{ij}^2 + \frac{G}{a_{ij}}\right)$$

i.e.

$$V_{tot} = -\sum_{i<j} m_i m_j\left(\frac{\omega^2}{2M}a_{ij}^2 + \frac{G}{a_{ij}}\right).$$

A mechanical system is in equilibrium if its total potential energy has an extremum. In our case the total potential energy V_{tot} is a sum of three terms. Each of them is proportional to:

$$f(a) = \frac{\omega^2}{2M}a^2 + \frac{G}{a}.$$

The extrema of this function can be found by taking its derivative with respect to a and requiring this derivative to be zero. We get:

$$\frac{\omega^2}{M}a - \frac{G}{a^2} = 0.$$

It leads to:

$$\omega^2 a^3 = GM \quad\text{or}\quad \omega^2 a^3 = G(m_1 + m_2 + m_3).$$

We see that all the terms in V_{tot} have extrema at the same values of $a_{ij} = a$. (In addition, the values of a and ω should obey the relation written above.) It is easy to show that it is a maximum. Thus, the quantity V_{tot} has a maximum at $a_{ij} = a$.

This means that our three masses can remain in fixed distances only if these distances are equal to each other:

$$a_{12} = a_{23} = a_{13} = a$$

and if the relation

$$\omega^2 a^3 = GM ,$$

where M the total mass of the system, holds. We have obtained the conditions (9) and (11) again.

THIRD METHOD

Let us consider again the point P_1 with the mass m_1 and the forces $\mathbf{F}_{21}$ and $\mathbf{F}_{31}$ given by the formulae (3) and (4). It follows from the text of the problem that the total moment (with respect to any fixed point or with respect to the mass center) of the forces acting on the point P_1 must be equal to zero. Thus, we have:

$$\mathbf{F}_{21}\times\mathbf{r}_1 + \mathbf{F}_{31}\times\mathbf{r}_1 = 0$$

where the symbol $\times$ denotes the vector product. Therefore

$$G\frac{m_1 m_2}{a_{12}^3}(\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{r}_1 + G\frac{m_1 m_3}{a_{13}^3}(\mathbf{r}_3 - \mathbf{r}_1) \times \mathbf{r}_1 = 0.$$

But

$$\mathbf{r}_1 \times \mathbf{r}_1 = 0.$$

Thus:

$$\frac{m_2}{a_{12}^3}\mathbf{r}_2 \times \mathbf{r}_1 + \frac{m_3}{a_{13}^3}\mathbf{r}_3 \times \mathbf{r}_1 = 0.$$

Using the formula (8), the last relation can be written as follows:

$$\frac{1}{a_{12}^3}(-m_1\mathbf{r}_1 - m_3\mathbf{r}_3) \times \mathbf{r}_1 + \frac{m_3}{a_{13}^3}\mathbf{r}_3 \times \mathbf{r}_1 = 0,$$

$$-\frac{m_3}{a_{12}^3}\mathbf{r}_3 \times \mathbf{r}_1 + \frac{m_3}{a_{13}^3}\mathbf{r}_3 \times \mathbf{r}_1 = 0,$$

$$\left(\frac{1}{a_{13}^3} - \frac{1}{a_{12}^3}\right)\mathbf{r}_3 \times \mathbf{r}_1 = 0.$$

The vectors $\mathbf{r}_1$ and $\mathbf{r}_3$ are non-collinear (and different from 0). Therefore

$$\mathbf{r}_3 \times \mathbf{r}_1 \neq 0$$

and

$$\frac{1}{a_{13}^3} - \frac{1}{a_{12}^3} = 0,$$

hence,

$$a_{12} = a_{13}.$$

Similarly, one gets:

$$a_{12} = a_{23} \ (= a).$$

We have re-derived the condition (11).

Taking into account that all the distances a_{ij} have the same value a, from the equation (7) concerning the point P$_1$, using the relation (2) we obtain:

$$G\frac{m_1 m_2}{a^3}(\mathbf{r}_2 - \mathbf{r}_1) + G\frac{m_1 m_3}{a^3}(\mathbf{r}_3 - \mathbf{r}_1) + m_1\omega^2\mathbf{r}_1 = 0,$$

$$-\left(G\frac{m_1}{a^3} + G\frac{m_2}{a^3}G\frac{m_3}{a^3}\right)m_1\mathbf{r}_1 + m_1\omega^2\mathbf{r}_1 = 0,$$

$$\frac{GM}{a^3} = \omega^2.$$

This is the condition (9). The same condition is got in result of similar calculations for the points P$_2$ and P$_3$.

The method described here does not differ essentially from the first method. In fact they are slight modifications of each other. However, it is interesting to notice how application of a proper mathematical language, e.g. the vector product, simplifies the calculations.

The relation (9) can be called a "generalized Kepler's law" as, in fact, it is very similar to the Kepler's law but with respect to the many-body system. As far as I know this generalized Kepler's law was presented for the first time right at the 20th IPhO.

Marking scheme

1. the proof that $\omega = \text{const}$ 1 point
2. the conditions at the equilibrium (conditions for the forces
 and their moments or extremum of the total potential energy)   3 points
3. the proof of the relation $a_{ij} = a$ 4 points
4. the proof of the relation $\omega^2 a^3 = GM$ 2 points

Remarks and typical mistakes in the pupils' solutions

No type of error was observed as predominant in the pupils' solutions. Practically all the mistakes can be put down to the students' scant experience in calculations and general lack of skill. Several students misunderstood the text of the problem and attempted to prove that the three masses should be equal. Of course, this was impossible. Moreover, it was pointless, since the masses were given. Almost all the participants tried to solve the problem by analyzing equilibrium of forces and/or their moments. Only one student tried to solve the problem by looking for a minimum of the total potential energy (unfortunately, his solution was not fully correct). Several participants solved the problem using a convenient reference system: one mass in the origin and one mass on the *x*-axis. One of them received a special prize.

Problem 3

The problem concerns investigation of transforming the electron microscope with magnetic guiding of the electron beam (which is accelerated with the potential difference $U = 511$ kV) into a proton microscope (in which the proton beam is accelerated with the potential difference $-U$). For this purpose, solve the following two problems:

A. An electron after leaving a device, which accelerated it with the potential difference U, falls into a region with an inhomogeneous field B generated with a system of stationary coils $L_1, L_2, \ldots, L_n$. The known currents in the coils are $i_1, i_2, \ldots, i_n$, respectively.

What should the currents $i_1', i_2', \ldots, i_n'$ in the coils $L_1, L_2, \ldots, L_n$ be, in order to guide the proton (initially accelerated with the potential difference $-U$) along the same trajectory (and in the same direction) as that of the electron?

HINT: The problem can be solved by finding a condition under which the equation describing the trajectory is the same in both cases. It may be helpful to use the relation:

$$\boldsymbol{p}\,\frac{d}{dt}\boldsymbol{p} = \frac{1}{2}\frac{d}{dt}\boldsymbol{p}^2 = \frac{1}{2}\frac{d}{dt}p^2.$$

B. How many times would the resolving power of the above microscope increase or decrease if the electron beam were replaced with the proton beam? Assume that the resolving power of the microscope (i.e. the smallest distance between two point objects whose circular images can be just separated) depends only on the wave properties of the particles.

Assume that the velocities of the electrons and protons before their acceleration are zero, and that there is no interaction between own magnetic moment of either electrons or protons and the magnetic field. Assume also that the electromagnetic radiation emitted by the moving particles can be neglected.

NOTE: Very often physicists use 1 electron-volt (1 eV), and its derivatives such as 1 keV or 1 MeV, as a unit of energy. 1 electron-volt is the energy gained by the electron that passed the potential difference equal to 1 V.

Perform the calculations assuming the following data:

Rest energy of electron: $E_e = m_e c^2 = 511$ keV
Rest energy of proton: $E_p = m_p c^2 = 938$ MeV

Solution

PART A

At the beginning one should notice that the kinetic energy of the electron accelerated with the potential difference $U = 511$ kV equals to its rest energy E_0. Therefore, at least in the case of the electron, the laws of the classical physics cannot be applied. It is necessary to use relativistic laws.

The relativistic equation of motion of a particle with the charge e in the magnetic field **B** has the following form:

$$\frac{d}{dt}\mathbf{p} = \mathbf{F}_L$$

where $\mathbf{p} = m_0 \gamma \mathbf{v}$ denotes the momentum of the particle (vector) and

$$\mathbf{F}_L = e\mathbf{v} \times \mathbf{B}$$

is the Lorentz force (its value is evB and its direction is determined with the right hand rule). m_0 denotes the (rest) mass of the particle and v denotes the velocity of the particle. The quantity γ is given by the formula:

$$\gamma = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

The Lorentz force $\mathbf{F}_L$ is perpendicular to the velocity **v** of the particle and to its momentum $\mathbf{p} = m_0 \gamma \mathbf{v}$. Hence,

$$\mathbf{F}_L \cdot \mathbf{v} = \mathbf{F}_L \cdot \mathbf{p} = 0 .$$

Multiplying the equation of motion by **p** and making use of the hint given in the text of the problem, we get:

$$\frac{1}{2}\frac{d}{dt}p^2 = 0 .$$

It means that the value of the particle momentum (and the value of the velocity) is constant during the motion:

$$p = m_0 v \gamma = \text{const}; \qquad v = \text{const}.$$

The same result can be obtained without any formulae in the following way:

The Lorentz force $\mathbf{F}_L$ is perpendicular to the velocity $\mathbf{v}$ (and to the momentum p as $\mathbf{p} = m_0\gamma\mathbf{v}$) and, as a consequence, to the trajectory of the particle. Therefore, there is no force that could change the component of the momentum tangent to the trajectory. Thus, this component, whose value is equal to the length of $\mathbf{p}$, should be constant: $\mathbf{p} = \text{const}$. (The same refers to the component of the velocity tangent to the trajectory as $\mathbf{p} = m_0\gamma\mathbf{v}$).

Let s denotes the path passed by the particle along the trajectory. From the definition of the velocity, we have:

$$\frac{ds}{dt} = v.$$

Using this formula, we can rewrite the equation of motion as follows:

$$v\frac{d}{ds}\mathbf{p} = \frac{ds}{dt}\frac{d}{ds}\mathbf{p} = \frac{d}{dt}\mathbf{p} = \mathbf{F}_L ,$$

$$\frac{d}{ds}\mathbf{p} = \frac{\mathbf{F}_L}{v} .$$

Dividing this equation by p and making use of the fact that $p = \text{const}$, we obtain:

$$v\frac{d}{ds}\frac{\mathbf{p}}{p} = \frac{\mathbf{F}_L}{vp}$$

and hence

$$\frac{d}{ds}\mathbf{t} = \frac{\mathbf{F}_L}{vp}$$

where $\mathbf{t} = \mathbf{p}/p = \mathbf{v}/v$ is the versor (unit vector) tangent to the trajectory. The above equation is exactly the same for both electrons and protons if and only if the vector quantity:

$$\frac{\mathbf{F}_L}{vp}$$

is the same in both cases.

Denoting corresponding quantities for protons with the same symbols as for the electrons, but with primes, one gets that the condition, under which both electrons and protons can move along the same trajectory, is equivalent to the equality:

$$\frac{\mathbf{F}_L}{vp} = \frac{\mathbf{F}'_L}{v'p'} .$$

However, the Lorentz force is proportional to the value of the velocity of the particle, and the directions of any two vectors of the following three: $\mathbf{t}$ (or $\mathbf{v}$), $\mathbf{F}_L$, $\mathbf{B}$ determine the direction of the third of them (right hand rule). Therefore, the above condition can be written in the following form:

$$\frac{e\mathbf{B}}{p} = \frac{e'\mathbf{B}'}{p'} .$$

Hence,

$$\mathbf{B}' = \frac{e}{e'}\frac{p'}{p}\mathbf{B} = \frac{p'}{p}\mathbf{B}.$$

This means that at any point the direction of the field **B** should be conserved, its orientation should be changed into the opposite one, and the value of the field should be multiplied by the same factor p'/p. The magnetic field **B** is a vector sum of the magnetic fields of the coils that are arbitrarily distributed in the space. Therefore, each of this fields should be scaled with the same factor $-p'/p$. However, the magnetic field of any coil is proportional to the current flowing in it. This means that the required scaling of the fields can only be achieved by the scaling of all the currents with the same factor $-p'/p$:

$$i'_n = -\frac{p'}{p}i_n.$$

Now we shall determine the ratio p'/p. The kinetic energies of the particles in both cases are the same; they are equal to $E_k = e|U| = 511$ keV. The general relativistic relation between the total energy E of the particle with the rest energy E_0 and its momentum p has the following form:

$$E^2 = E_0^2 + p^2c^2$$

where c denotes the velocity of light.

The total energy of considered particles is equal to the sum of their rest and kinetic energies:

$$E = E_0 + E_k.$$

Using these formulae and knowing that in our case $E_k = e|U| = E_e$, we determine the momenta of the electrons (p) and the protons (p'). We get:

a) electrons:

$$(E_e + E_e)^2 = E_e^2 + p^2c^2,$$

$$p = \frac{E_e}{c}\sqrt{3}.$$

b) protons

$$(E_p + E_e)^2 = E_p^2 + p'^2c^2,$$

$$p' = \frac{E_e}{c}\sqrt{\left(\frac{E_p}{E_e}+1\right)^2 - \left(\frac{E_p}{E_e}\right)^2}.$$

Hence,

$$\frac{p'}{p} = \frac{1}{\sqrt{3}}\sqrt{\left(\frac{E_p}{E}+1\right)^2 - \left(\frac{E_p}{E_e}\right)^2} \approx 35.0$$

and

$$i'_n = -35.0\,i_n.$$

It is worthwhile to notice that our protons are 'almost classical', because their kinetic energy $E_k(=E_e)$ is small compared to the proton rest energy E_p. Thus, one can expect that the momentum of the proton can be determined, with a good accuracy, from the classical considerations. We have:

$$E_e = E_k = \frac{p'^2}{2m_p} = \frac{p'^2 c^2}{2m_p c^2} = \frac{p'^2 c^2}{2E_p},$$

$$p' = \frac{1}{c}\sqrt{2E_e E_p}.$$

On the other hand, the momentum of the proton determined from the relativistic formulae can be written in a simpler form since $E_p/E_e \gg 1$. We get:

$$p' = \frac{E_e}{c}\sqrt{\left(\frac{E_p}{E_e}+1\right)^2 - \left(\frac{E_p}{E_e}\right)^2} = \frac{E_e}{c}\sqrt{2\frac{E_p}{E_e}+1} \approx \frac{E_e}{c}\sqrt{2\frac{E_p}{E_e}} = \frac{1}{c}\sqrt{2E_e E_p}.$$

In accordance with our expectations, we have obtained the same result as above.

PART B

The resolving power of the microscope (in the meaning mentioned in the text of the problem) is proportional to the wavelength, in our case to the length of the de Broglie wave:

$$\lambda = \frac{h}{p}$$

where h denotes the Planck constant and p is the momentum of the particle. We see that λ is inversely proportional to the momentum of the particle. Therefore, after replacing the electron beam with the proton beam the resolving power will be changed by the factor $p/p' \approx 1/35$. It means that our proton microscope would allow observation of the objects about 35 times smaller than the electron microscope.

Marking scheme

1. the relativistic equation of motion	1 point
2. independence of p and v of the time	1 point
3. identity of $e\mathbf{B}/p$ in both cases	2 points
4. scaling of the fields and the currents with the same factor	1 point
5. determination of the momenta (relativistically)	1 point
6. the ratio of the momenta (numerically)	1 point
7. proportionality of the resolving power to λ	1 point
8. inverse proportionality of λ to p	1 point
9. scaling of the resolving power	1 point

Remarks and typical mistakes in the pupils' solutions

Some of the participants tried to solve the problem by using laws of classical mechanics only. Of course, this approach was entirely wrong. Some students tried to find the required condition by equating "accelerations" of particles in both cases. They understood the "acceleration" of the particle as a ratio of the force acting on the particle to the "relativistic" mass of the particle. This approach is incorrect. First, in relativistic physics the relationship between force and acceleration is more complicated. It deals with not one "relativistic" mass,

but with two "relativistic" masses: transverse and longitudinal. Secondly, identity of trajectories need not require equality of accelerations.

The actual condition, i.e. the identity of $e\mathbf{B}/p$ in both cases, can be obtained from the following two requirements:

1° in any given point of the trajectory the curvature should be the same in both cases;

2° in the vicinity of any given point the plane containing a small arc of the trajectory should be oriented in space in both cases in the same way.

Most of the students followed the approach described just above. Unfortunately, many forgot about the second requirement (they neglected the vector character of the quantity $e\mathbf{B}/p$).

EXPERIMENTAL PROBLEM[1]

The following equipment is provided:

1. Two piezoelectric discs of thickness 10 mm with evaporated electrodes (Fig. 4.1) fixed in holders on the jaws of the calipers;
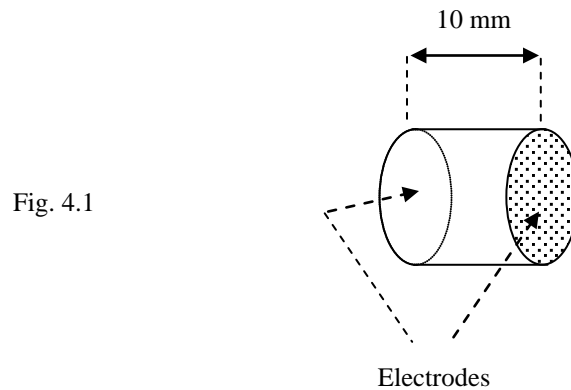

Fig. 4.1

Electrodes

2. The calibrated sine wave oscillator with a photograph of the control panel, explaining the functions of the switches and regulators;
3. A double channel oscilloscope with a photograph of the control panel, explaining the functions of the switches and regulators;
4. Two closed plastic bags containing liquids;
5. A beaker with glycerin (for wetting the discs surfaces to allow better mechanical coupling);
6. Cables and a three way connector;
7. A stand for support the bags with the liquids;
8. Support and calipers.

A piezoelectric material changes its linear dimensions under the influence of an electric field and vice-versa, the distortion of a piezoelectric material induces an electrical field. Therefore, it is possible to excite the mechanical vibrations in a piezoelectric material by applying an alternating electric field, and also to induce an alternating electric field by mechanical vibrations.

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[1] The Organizing Committee planned to give another experimental problem: a problem on high T_c superconductivity. Unfortunately, the samples of superconductors, prepared that time by a factory, were of very poor quality. Moreover, they were provided after a long delay. Because of that the organizers decided to use this problem, which was also prepared, but considered as a second choice.

A. Knowing that the velocity of longitudinal ultrasonic waves in the material of the disc is about $4 \cdot 10^3$ m/s, estimate roughly the resonant frequency of the mechanical vibrations parallel to the disc axis. Assume that the disc holders do no restrict the vibrations. (Note that other types of resonant vibrations with lower or higher frequencies may occur in the discs.)

Using your estimation, determine experimentally the frequency for which the piezoelectric discs work best as a transmitter-receiver set for ultrasound in the liquid. Wetting surfaces of the discs before putting them against the bags improves penetration of the liquid in the bag by ultrasound.

B. Determine the velocity of ultrasound for both liquids without opening the bags and estimate the error.

C. Determine the ratio of the ultrasound velocities for both liquids and its error.

Complete carefully the synopsis sheet. Your report should, apart from the synopsis sheet, contain the descriptions of:

- method of resonant frequency estimation;
- methods of measurements;
- methods of estimating errors of the measured quantities and of final results.

Remember to define all the used quantities and to explain the symbols.

<table>
<tr>
<td colspan="4" align="center">**Synopsis Sheet**[1]</td>
</tr>
<tr>
<td rowspan="2">**A**</td>
<td colspan="3">Formula for estimating the resonant frequency: Results (with units):</td>
</tr>
<tr>
<td colspan="3">Measured best transmitter frequency (with units):   Error:</td>
</tr>
<tr>
<td rowspan="5">**B**</td>
<td colspan="3">Definition of measured quantity: Symbol: Results: Error:</td>
</tr>
<tr>
<td colspan="3">Final formula for ultrasound velocity in liquid:</td>
</tr>
<tr>
<td colspan="3" align="center">Velocity of ultrasound (with units):   Error:</td>
</tr>
<tr>
<td colspan="3">Liquid A
Liquid B</td>
</tr>
<tr>
<td colspan="3">Ratio of velocities: Error:</td>
</tr>
</table>

Solution (draft)[1]

---

[1] In the real Synopsis Sheet the students had more space for filling.

A. As the holders do not affect vibrations of the disc we may expect antinodes on the flat surfaces of the discs (Fig. 4.2; geometric proportions not conserved). One of the frequencies is expected for

$$l = \tfrac{1}{2}\lambda = \frac{v}{2f},$$

where v denotes the velocity of longitudinal ultrasonic wave (its value is given in the text of the problem), f - the frequency and l - the thickness of the disc. Thus:

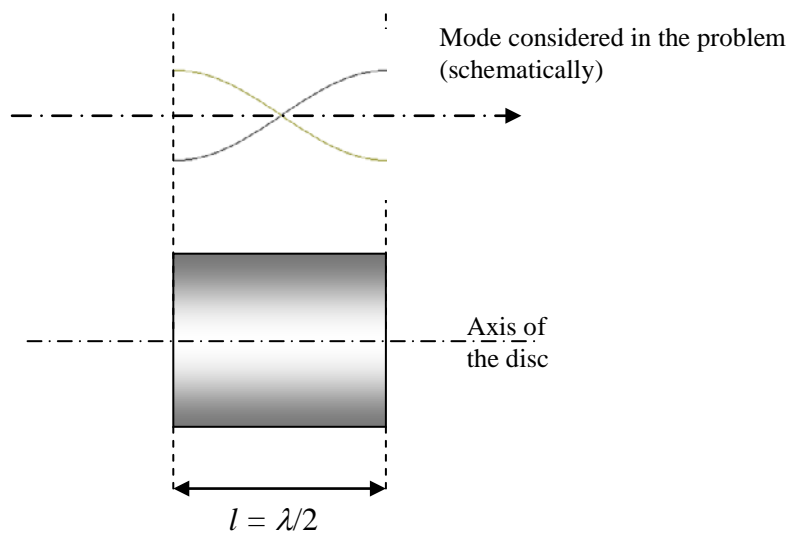$$f = \frac{v}{2l}.$$

Numerically $f = 2 \cdot 10^5$ Hz = 200 kHz.

Mode considered in the problem (schematically)

Axis of the disc

$l = \lambda/2$

Fig. 4.2

One should stress out that different modes of vibrations can be excited in the disc with height comparable to its diameter. We confine our considerations to the modes related to longitudinal waves moving along the axis of the disc as the sound waves in liquids are longitudinal. We neglect coupling between different modes and require antinodes exactly at the flat parts of the disc. We assume also that the piezoelectric effect does not affect velocity of ultrasound. For these reasons the frequency just determined should be treated as only a rough approximation. However, it indicates that one should look for the resonance in vicinity of 200 kHz.

The experimental set-up is shown in Fig. 4.3. The oscillator (generator) is connected to one of the discs that works as a transmitter and to one channel of the oscilloscope. The second disc is connected to the second channel of the oscilloscope and works as a receiver. Both discs are placed against one of the bags with liquid (Fig. 4.4). The distance d can be varied.

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[1] This draft solution is based on the camera-ready text of the more detailed solution prepared by Dr. Andrzej Kotlicki and published in the proceedings [3]
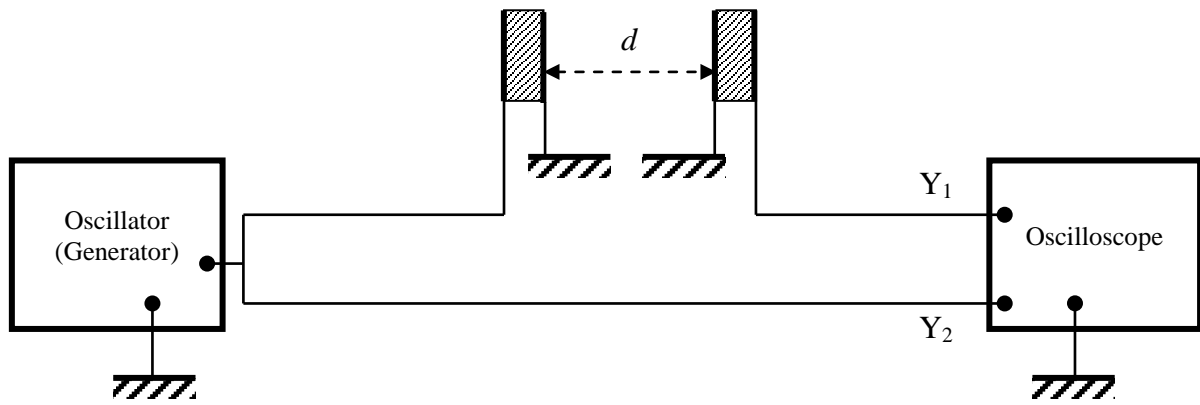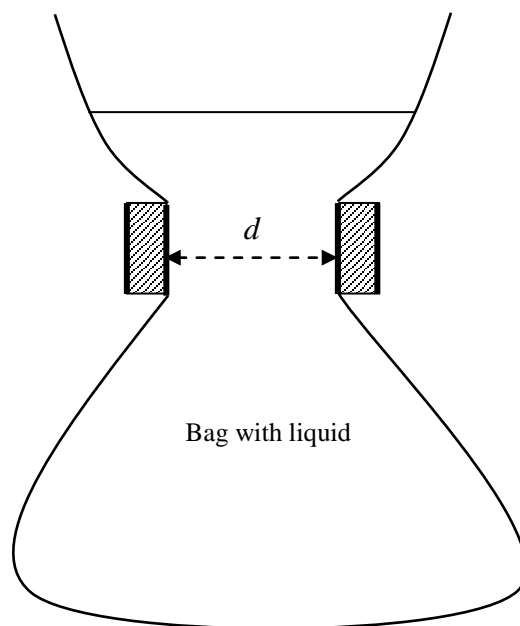
Fig. 4.3

Fig. 4.4

One searches for the resonance by slowly changing the frequency of the oscillator in the range 100 – 1000 kHz and watching the signal on the oscilloscope. In this way the students could find a strong resonance at frequency $f \approx 220$ kHz. Other resonance peaks could be found at about 110 kHz and 670 kHz. They should have been neglected as they are substantially weaker. (They correspond to some other modes of vibrations.) Accuracy of these measurements was 10 kHz (due to the width of the resonance and the accuracy of the scale on the generator).

B. The ultrasonic waves pass through the liquid and generate an electric signal in the receiver. Using the same set-up (Fig. 4.3 and 4.4) we can measure dependence of the phase shift between the signals at Y_1 and Y_2 vs. distance between the piezoelectric discs d at the constant frequency found in point **A**. This phase shift is $\Delta\varphi = 2\pi df / v_l + \varphi_0$, where v_l denotes velocity of ultrasound in the liquid. φ_0 denotes a constant phase shift occurring when ultrasound passes trough the bag walls (possibly zero). The graph representing dependence $d(\Delta\varphi)$ should be a straight line. Its slope allows to determine v_l and its error. In general, the

measurements of $\Delta\varphi$ are difficult for many reflections in the bag, which perturb the signal. One of the best ways is to measure d only for $\Delta\varphi = n\pi$ (n - integer) as such points can be found rather easy. Many technical details concerning measurements can be found in [3] (pp. 37 and 38).

The liquids given to the students were water and glycerin. In the standard solution the author of the problem received the following values:

$$v_{\text{water}} = (1.50 \pm 0.10) \cdot 10^3 \text{ m/s}; \quad v_{\text{glycerin}} = (1.96 \pm 0.10) \cdot 10^3 \text{ m/s}.$$

The ratio of these values was 1.31 ± 0.15.

The ultrasonic waves are partly reflected or scattered by the walls of the bag. This effect somewhat affects measurements of the phase shift. To minimize its role one can measure the phase shift (for a given distance) or distance (at the same phase shift) several times, each time changing the shape of the bag. As regards errors in determination of velocities it is worth to mention that the most important factor affecting them was the error in determination of the frequency. This error, however, practically does not affect the ratio of velocities.

Marking Scheme

Frequency estimation

1. Formula   1 point
2. Result (with units)   1 point
3. Method of experimental determining the resonance frequency   1 point
4. Result (if within 5% of standard value)   2 points
5. Error   1 point

Measurements of velocities

1. Explanation of the method   2 points
2. Proper number of measurements in each series   3 points
3. Result for velocity in the first liquid (if within 5% of standard value)   2 points
4. Error of the above   1 point
5. Result for velocity in the second liquid (if within 5% of standard value) 2 points
6. Error of the above   1 point

Ratio of velocities

1. Result (if within 3% of standard value)   2 points
2. Error of the above   1 point

Typical mistakes

The results of this problem were very good (more than a half of competitors obtained more than 15 points). Nevertheless, many students encountered some difficulties in estimation of the frequency. Some of them assumed presence of nodes at the flat surfaces of the discs (this assumption is not adequate to the situation, but accidentally gives proper formula). In part B some students tried to find distances between nodes and antinodes for ultrasonic standing wave in the liquid. This approach gave false results as the pattern of standing waves in the bag is very complicated and changes when the shape of the bag is changed.

Acknowledgement

Literature

[1] **Waldemar Gorzkowski** and **Andrzej Kotlicki**, *XX Międzynarodowa Olimpiada Fizyczna - cz. I*, Fizyka w Szkole *nr* **1/90**, pp. 34 - 39

[2] **Waldemar Gorzkowski**, *XX Międzynarodowa Olimpiada Fizyczna - cz. II*, Fizyka w Szkole *nr* **2/3-90**, pp. 23 - 32

[3] *XX International Physics Olympiad - Proceedings of the XX International Physics Olympiad, Warsaw (Poland), July 16 - 24, 1989*, ed. by W. Gorzkowski, World Scientific Publishing Company, Singapore 1990 [ISBN 981-02-0084-6]