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Task 1.

(a) Assume the droplet reaches and maintains terminal velocity throughout, from $h = 4$ km to $h = 0$ km. This is valid because the droplet has a small mass compared to air resistance, and normally forms much higher than 4 km, so it has enough falling time to reach steady state velocity. Water and ice are assumed to be good conductors of heat, so most heat transfer occurs via conduction.

By Newton's Law of Cooling, $\frac{dQ}{dt} = k(T - T_{in})$ where T_{in} is internal temperature. However, since water and ice conduct heat well, and all heat transferred is used to melt/freeze the droplet, thus during the phase changes of melting and freezing, $T_{in} = 0^\circ C$ constantly. Also, since evaporation/condensation are neglected, size and surface area of droplet remain constant. Furthermore, density of atmosphere is constant so all this means we can assume k is constant.

Thus $\frac{dQ}{dt} = kT$

$$Q = \int dQ = \int kT\,dt$$

$$\frac{dh}{dt} = v$$

$$\Rightarrow Q = \int \frac{kT}{v}\,dh = \frac{k}{v}\int T\,dh$$

All heat transferred goes towards latent heat for both melting ($1.5 \le h \le 4$) and freezing ($0 \le h \le 1.5$). $Q = mL$ thus $m \propto \int T\,dh$.

Thus mass fraction $\eta = \dfrac{-\int_0^{1.5} T\,dh}{\int_{1.5}^{4} T\,dh} = \dfrac{\frac{1}{2}(2.5)(4) - \frac{1}{2}(1)(2)}{\frac{1}{2}(2.5)(4)} = \dfrac{4}{5}$

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Task 1(b)

Let h be in km, T be in °C.

Now the ice droplet melts until a point h_0, after which all heat absorbed is used to increase the temperature of the droplet (which is now fully water).

From the graph, $T = 8 - \frac{h}{500}$

Same amount of latent heat must be absorbed to melt the ice fully:

$$\frac{1}{2}\left(4 - \frac{h_0}{1000}\right)\left(8 - 2\frac{h_0}{1000}\right) = \frac{1}{2}(2.5)(4)$$

$$2\left(\frac{h_0}{1000}\right)^2 - 16\frac{h_0}{1000} + 22 = 0$$

$$h_0 = 6236\,m \text{ (reject since } h_0 < 2)$$

$$\text{or } \underline{\underline{1763.9\,m}}$$

$$dQ = \frac{k}{v}(-dh)(T - T_{in})$$

(We have to be careful of the sign of dh which is negative)

Also, after all ice has melted, all heat is used to raise the temperature T_{in}.

$$dQ = mc_w dT_{in}$$

$$mc_w dT_{in} = \frac{k}{v}(T - T_{in})(-dh)$$

We know that $\frac{k}{v}\frac{1}{2}(2500)(4) = mL$ is the latent heat of melting, so

$$\frac{k}{v} = \frac{mL}{5000}$$

$$\frac{dT_{in}}{dh} = -\frac{L}{5000c_w}\left(8 - \frac{h}{500} - T_{in}\right)$$

From here on, for ease of notation, let T refer to T_{in} rather than temperature of air.

Let $a = -\frac{L}{5000c_w}$.

$$\frac{dT}{dh} = a\left(8 - \frac{h}{500} - T\right)$$

$$\frac{dT}{dh} + aT = 8a - \frac{a}{500}h \quad \text{(integrating factor} = e^{ah})$$

$$Te^{ah} = \int \left(8a - \frac{a}{500}h\right)e^{ah}\,dh \quad \text{(integrate by parts)}$$

$$= \left(8a - \frac{a}{500}h\right)\frac{1}{a}e^{ah} + \int \frac{a}{500}\frac{1}{a}e^{ah}\,dh$$

$$= \left(8 - \frac{h}{500}\right)e^{ah} + \frac{1}{500a}e^{ah} + c, \quad c: \text{constant}$$

$$T = 8 + \frac{1}{500a} - \frac{h}{500} + ce^{-ah}$$

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Task (cb) (continued)

$$T = 8 + \frac{1}{500a} - \frac{h}{500} + ce^{-ah}$$

When $h = h_0$, $T = 0$: $c = \dfrac{-\left(8 + \frac{1}{500a} - \frac{h_0}{500}\right)}{e^{-ah_0}}$

$$\therefore T = 8 + \frac{1}{500a} - \frac{h}{500} - \left(8 + \frac{1}{500a} - \frac{h_0}{500}\right) e^{a(h_0 - h)}$$

Sub. $a = -\dfrac{L}{5000c_w} = -\dfrac{334 \times 10^3}{5000 \times 4200}\ \mathrm{m}^{-1}$ and $h_0 = 1763.9\ \mathrm{m}$ and $h = 0$

to get $T_{ground} = \boxed{7.87^\circ C}$, which is just below $T_{air} = 8^\circ C$.