The 3[nd] Gulf Physics Olympiad — Theoretical Competition

Muscat, Oman — October 7th 2019

- The examination lasts for 5 hours. There are 3 problems worth in total 30 points. **Please note that the point values of the three theoretical problems are not equal.**
- **You must not open the envelope with the problems before the signal of the beginning of competition.**
- **You are not allowed to leave your working place without permission.** If you need any assistance (broken calculator, need to visit a restroom, etc), please raise your hand until an organizer arrives.
- Use only the front side of the sheets of paper.
- For each problem, there are **dedicated Solution Sheets** (see header for the number and pictogram). Write your solutions onto the appropriate Solution Sheets. For each Problem, the Solution Sheets are numbered; use the sheets according to the enumeration. **Always mark which Problem Part and Question you are dealing with.** Copy the final answers into the appropriate boxes of the **Answer Sheets**. There are also **Draft** papers; use these for writing things which you don't want to be graded. If you have written something that you don't want to be graded onto the Solution Sheets (such as initial and incorrect solutions), cross these out.
- If you need more paper for a certain problem, please raise your hand and tell an organizer the problem number; you are given two Solution sheets (you can do this more than once).
- **You should use as little text as possible:** try to explain your solution mainly with equations, numbers, symbols and diagrams. Though in some places textual explanation may be unavoidable.
- **After the signal signifying the end of examination you must stop writing immediately.** Put all the papers into the envelope at your desk. **You are not allowed to take any sheet of paper out of the room.** If you have finished solving before the final sound signal, please raise your hand.

Problem T1. Zero gravity *(10 points)*

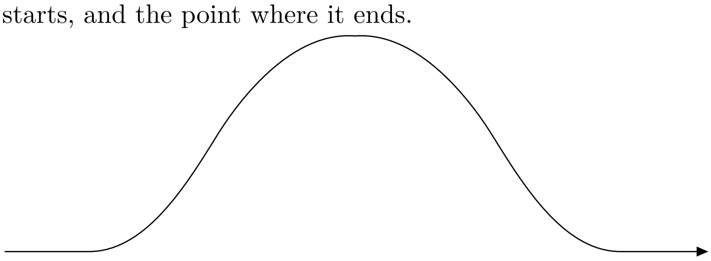
In all your subsequent calculations, you may use the following physical constants and their numerical values.
The radius of Earth $R_\oplus = 6.4 \times 10^6\,\mathrm{m}$.
Free fall acceleration at the sea level $g = 9.81\,\mathrm{m/s^2}$.

Part A. Zero-g flight *(3 points)*

Astronauts can experience weightlessness during their spaceflight. However, there is a cheaper way to experience weightlessness other than boarding a spaceship: there are airplanes specifically designed to create weigthlessness on board during a certain time period. Such an airplane is shown in the photo.

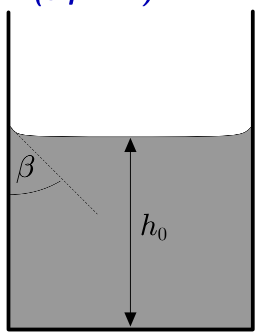
For this Part, the following values can be also used.
The speed of sound at the flying altitude $c_s = 300\,\mathrm{m/s}$.
The altitude (from the sea level) at which the airplane starts zero-g-flight (the flight segment during which the objects on board the aircraft have zero weight) is $h_0 = 7600\,\mathrm{m}$.
The speed of the airplane when it starts zero-g-flight is $v_0 = 460\,\mathrm{km/h}$.
The angle between the horizontal plane and the direction of the velocity vector at the moment when the airplane starts zero-g-flight $\alpha_0 = 47°$.

i. (0.5 pts) Below is a sketch of a zero-g flight trajectory (the one providing the longest duration of weightlessness might be slightly different). Mark on it the point where zero-g flight starts, and the point where it ends.



ii. (0.5 pts) What should be the direction and magnitude of the acceleration of the airplane to ensure that the passengeres would feel weightlessness?
iii. (0.5 pts) What is the speed of the airplane at the highest point of its trajectory?
iv. (0.5 pts) How long does it take for the airplane to reach the highest point on its trajectory from the moment when it starts zero-g-flight?
v. (0.5 pts) What is the altitude of the airplane at the highest point of its trajectory from the sea level?
vi. (0.5 pts) The possible values of the initial speed v_0 and initial ascending angle α_0 are limited by the robustness of the airplane's construction, and by the maximal thrust provided by the engines; the numerical values given above can be considered to be optimal, i.e. yielding the longest period during which the passengers experience weightlessness. Assuming that there are no restrictions on the final diving angle (the angle between the horizontal plane and the direction of the velocity vector at the moment when the airplane ends zero-g-flight) while the only limitation on the speed is that it cannot be larger than the speed of sound, what is the maximal total duration of a zero-g flight segment?

Part B. Glass of water in weightlessness *(3 points)*



Consider a partially filled glass of water on board this aircraft. The glass is cylindrical, of radius $r = 3\,\mathrm{cm}$; the walls of the glass are negligibly thin. At the moment when the airplane starts zero-g flight, the water surface is flat except for the small meniscus of negligible height near the walls of the glass (see the figure depicting axial cross-section of the glass), and the depth of water is $h_0 = 3\,\mathrm{cm}$. The contact angle of the water in the glass (the angle between the tangent to the water surface and the surface of the glass at the point where the water surface and glass are in direct contact, see the figure) is $\beta = 0°$ (the figure is illustrative).

i. (1 pt) Under the condition of weightlessness, the water surface will take a new equilibrium shape. Sketch the shape of the water surface at the axial cross-section of the glass.
ii. (1 pt) What is the minimal distance between the water surface and the bottom of the glass at the new equilibrium state?
iii. (1 pt) Under normal conditions, this glass can hold up to $V_0 = 200\,\mathrm{ml}$ water. What is the maximal volume of water which can be held in this glass in weightlessness? Sketch also the corresponding shape of the water surface at the axial cross-section of the glass.

Part C. Sharpshooter on geostationary orbit *(4 points)*

Weightlessness can be experienced also on spaceships performing ballistic motion (motion when engines are switched off). Let us consider an astronaut on geostationary orbit. This is a circular orbit around Earth which lies in the equatorial plane, and the period of motion on which is equal to $T_0 = 24\,\mathrm{h}$.
i. (0.7 pts) What is the radius of the geostationary orbit?
ii. (1.8 pts) For research reasons, the astronaut wants to hit his own spaceship with a bullet fired from a rifle equipped onto the spaceship. The speed of the bullet leaving the rifle is $u_0 = 1200\,\mathrm{m/s}$, the bullet's velocity lies on the orbital plane. Under which angle with respect to the vector pointing towards the centre of the Earth does he needs to aim the rifle if he wants to hit the spaceship within the next 40 hours? You don't need to prove that there is only one suitable shooting angle.
You may use the expression for the total energy of an elliptical orbit, $E_\mathrm{total} = -\frac{GM_\oplus m}{2a}$, where a is the semi-major axis.
iii. (1.5 pts) He also tries out another rifle the bullet speed of which can be freely adjusted from zero to the maximal speed $u_m = 300\,\mathrm{m/s}$. With this rifle, he aims strictly along the motion of the spaceship. What is the bullet's smallest possible travel time until hitting the spaceship?

Problem T2. Controlled fusion *(11 points)*

In all your subsequent calculations, you may use the following physical constants and their numerical values.
Boltzmann constant $k_B = 1.381 \times 10^{-23}\,\mathrm{J}\cdot\mathrm{K}^{-1}$
Elementary charge $e = 1.602 \times 10^{-19}\,\mathrm{C}$
Electron's mass $m_e = 9.109 \times 10^{-31}\,\mathrm{kg}$
Planck's constant $h = 6.626 \times 10^{-34}\,\mathrm{m}^2\cdot\mathrm{kg}\cdot\mathrm{s}^{-1}$
Permittivity of free space $\epsilon_0 = 8.854 \times 10^{-12}\,\mathrm{F}\cdot\mathrm{m}^{-1}$

Nuclear fusion is a reaction where light atomic nuclei merge to form a larger nucleus. The difference in the rest energies of the fusing nuclei and the fusion product is released as heat. For instance, if deuterium (consisting of one neutron and one proton, denoted as D) and tritium (consisting of two neutrons and one proton, denoted as T) merge, they will form an α-particle, a neutron, and 14 MeV of energy. While humans have learned how to ignite fusion reaction explosively in hydrogen bombs, they are still struggling to succeed in *controlled fusion*, i.e. to control fusion reaction so that the released heat could be used for operating power plants. The most feasible reaction for a controlled fusion is the above mentioned D-T reaction which will be addressed by this Problem.

Part A. General considerations *(0.5 points)*

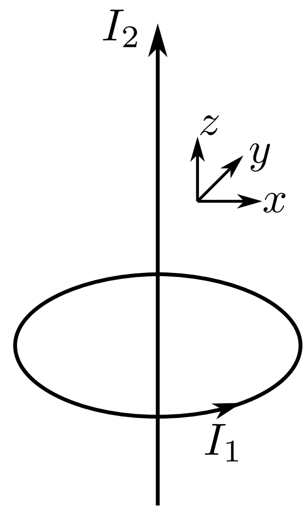
In what follows, we shall express the temperature in electron volts; this is a common practice for so high temperatures. 1 eV corresponds to such a temperature T by which the characteristic thermal energy k_BT equals to the potential energy of an electron in electrostatic potential of $V = 1\,\mathrm{V}$.

For a power plant, the released fusion energy must be larger than the total energy loss. It can be shown that for an optimally designed D-T reactor (device in which the controlled fusion takes place), the temperature of the deuterium and tritium nuclei should be $T_0 = 14\,\mathrm{keV}$ while the product of the number density of particles n (the number of particles per volume) and the *confinement time* τ (the time during which density n remains roughly constant) should not be less than $2 \times 10^{20}\,\mathrm{s/m}^3$; this requirement is know as the Lawson criterion. The main techonlogical challenge is to achieve a long enough confinement of the hot plasma.

i. (0.5 pts) Express the fusion temperature T_0 in Kelvins.

Part B. Tokamak *(2.5 points)*

The most popular design of fusion reactors is tokamak. In a tokamak, charged particles move along magnetic field lines and are confined because the field lines are confined into a finite volume of space. Qualitatively, the magnetic field lines have the same shape as in the case of an infinitely long straight current passing coaxially through a circular current loop. In the following subtasks, you're expected to provide the sketches in a 3d projection as shown in the figure.



i. (0.5 pts) Sketch magnetic field lines of a infinitely long straight current.

ii. (0.5 pts) Sketch magnetic field lines of a circular current loop.

iii. (0.75 pts) Sketch a magnetic field line of an infinitely long straight current passing coaxially through a circular current loop which starts from a small distance from the circular current.

iv. (0.75 pts) For the same current configuration as before, sketch a magnetic field line starting from a small distance from the straight current.

Part C. Cold fusion *(3.5 points)*

"Cold fusion" refers to a muon-catalytic fusion process by which an electron in a hydrogen molecule (which can include one deuterium and one tritium nucleus) is substituted by a muon. Muon, having a 207 times larger mass than an electron, brings the nuclei in the molecule closer to each other, thereby increasing the probability of their fusion. The idea of such a catalytic fusion was suggested in 1947-48 by A. Sakharov and F.C. Frank, and lead to a short-lived research boom in 1989 after an erraneous report of a successful fusion at room temperatures by M. Fleischmann and S. Pons. The problem with muon-catalytic fusion is that the energetic cost of producing one muon is larger than the total energy released by fusion reaction mediated by one muon; the possible solutions are either decreasing the energetic cost of a muon, or increasing the number of fusions mediated by a single muon. In what follows, we consider a simple approach to understand why substituting electrons with muons will decrease the size of an atom.

i. (1 pt) Using classical mechanics and considering an electron on a circular orbit of radius R around a point-like nucleus of charge $+2e$, relate the momentum p of the electron to the orbit's radius R.

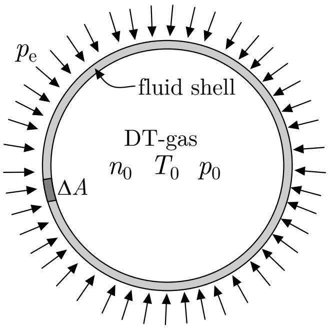
ii. (1 pt) At the ground state, the total energy is as small as possible; meanwhile, the state of the electron (muon) cannot violate the uncertainty principle. From these considerations, find an estimate for the radius R at the ground state.

iii. (1 pt) By the subtask i, we neglected the distance between the two nuclei in the molecule which was permissable for estimating the orbital radius R. Now, however, we want also to get an estimate for the distance between the nuclei. To that end, consider another simple model. The two electrons (muons) on their orbit form a ball-like cloud: let us assume that there is a spherical ball of radius R carrying a total charge $-2e$, homogeneously distributed over the entire volume of the ball. Inside the charged ball, there are two nuclei which can be considered as point masses and point charges (of charge $+e$ each), able to move frictionlessly inside the ball. Find the equilibrium distance d between the nuclei.

iv. (0.5 pts) Based on the model suggested above, by how many times is the distance between the deuterium and tritium atoms reduced when orbital electrons are substituted by muons?

Part D. Inertial confinement fusion *(4.5 points)*

Third approach to controlled fusion is based on the idea that due to mass and inertia, it takes some time, although short, for any hot blob of matter to explode and scatter. In order to satisfy the Lawson criterion one can inrease the confinement time, but one can also increase the number density n. In the *inertial confinement fusion* devices, powerful beams are used to create highly compressed balls of gas of densities exceeding the density of lead by hundreds of times. In what follows, we consider this approach by adopting a simple model: a liquid spherical shell of total mass M and radius r is surrounding a ball of gas of number density n_0, temperature T_0, and pressure $p_0 = k_B n_0 T_0$ (in reality, the shell is solid, but at really high pressures, solids essentially liquify); see the figure. Each gas molecule consists of a deuterium nucleus, tritium nucleus, and two electrons. The thickness of the walls of the spherical liquid shell δ is much smaller than r ($\delta \ll r$).



i. (0.5 pts) Consider a small piece of shell of surface area ΔA. Express its mass in terms of the quantities introduced above.

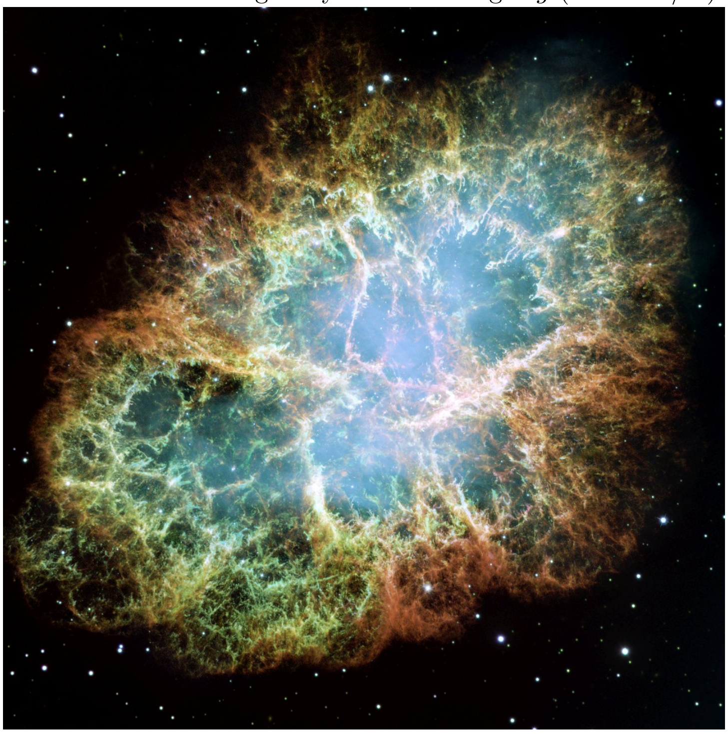
ii. (1 pt) External pressure p_e ($p_e \gg p_0$) is applied to the shell. Express the initial acceleration of a small piece of the shell in terms of the quantities introduced until now.

iii. (1.5 pts) While the shell contracts due to external pressure, the pressure inside grows, and at a certain moment, it becomes larger than the external pressure. Express the minimal radius of the shell r_m and the maximal temperature inside the shell T_m (which are achieved when the surrounding shell stops for a moment before reversing its direction of motion) in terms of the quantities introduced above. Keep in mind that the inside temperature becomes so high that the gas is converted into a completely ionized plasma made of nuclei and electrons. You may assume that p_e remains constant during the entire process (this might not be entirely true, but under this assumption, we shall still be able to get a correct order of magnitude for the answer), the shell contracts while retaining its spherical shape, and the thermal energy transferred to the shell can be neglected.

iv. (1.5 pts) The huge external pressure p_e is created by irradiating the shell from outside, isotropically from all sides, with a laser of total output power P. As a result, the outer layers of the shell are evaporated, and the evaporated atomic nulcei flow away at the average speed of u. Estimate the pressure p_e in terms of P, r, and u; you may assume that u is much smaller than the speed of light.

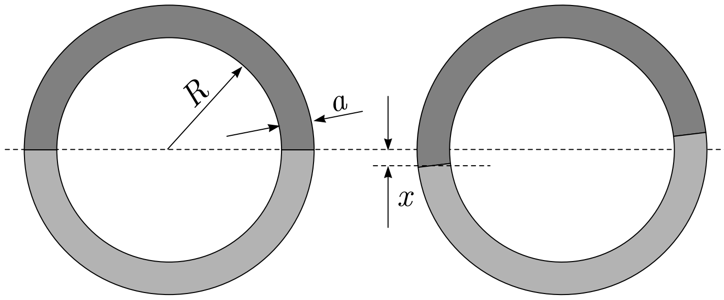
Problem T3. Rayleigh-Taylor instability *(9 points)*

Lord Rayleigh showed in 1883 that a layer of dense liquid on top of a layer of less dense liquid is unstable: even if the interface between the two liquids is initially perfectly flat and horizontal, small perturbations in the interface shape grow exponentially in time: at some places, heavy liquid starts to flow down displacing light liquid beneath, and in other places, light liquid starts flowing up — this phenomenon is nowadays known as the Rayleigh-Taylor instability. It plays an important role in many fields of physics. For instance, following a supernova explosion, shock waves of dense plasma decelerate due to "eating up" the regions with less dense plasma. This means that in the frame of reference of the decelerating shock wave, the force of inertia is pointing in the direction of the shock wave propagation. The direction of the force of inertia defines the "down" direction, so that the more dense plasma of the shock wave appears to be "atop" the less dense plasma of the interstellar space. Late (nonlinear) stages of the instability are characterized by fascinating filamentary structures, see the image of the Crab nebula below. In techonology, Rayleigh-Taylor instability can be undesirable and for instance, makes it very difficult to accomplish the inertial confinement fusion project: when initially an almost perfectly round sphere is being compressed, it becomes irregularly distorted — like an empty can of Coke when you try to compress it. In what follows, we construct mathematically simple models to shed insight into the physics of the Rayleigh–Taylor instability. Assume everywhere below that there is a downwards gravity field of strength g $(= 9.81\,\mathrm{m/s^2})$.



Part A. Instability growth rate *(4 points)*

i. (1 pt) Consider a circular O-tube the lower half of which is filled with a liquid of density ρ_1, and upper half — with a liquid of density $\rho_2 > \rho_1$, see the figure. Let the radius of the circle R be much larger than the diameter of the tube a (neglect the wall thickness). When the interfaces at the both sides of the O-tube are exactly at the same level (the left sketch), the system is at equilibrium. By how much will the potential energy of the system change when the interface in the left part of the tube is lowered by x (as shown in the sketch on right)? Express the answer in terms of the quantities introduced above. Here and in what follows, assume that $x \ll R$ and use the resulting approximations.



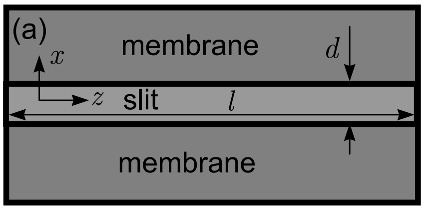
ii. (1 pt) Suppose now that the system will evolve by itself starting from the position shown in the right sketch, let us denote the speed with which the interface in the tube moves with $v = \frac{\mathrm{d}x}{\mathrm{d}t}$. Express the kinetic energy of the system in terms of v and the other quantities defined above.

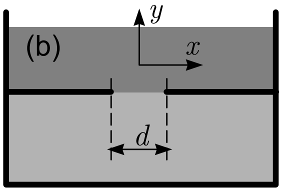
iii. (1 pt) Show that the acceleration of the interface is proportional to its displacement x by taking a time derivative of the energy conservation law, and that the displacement can grow exponentially in time so that x is proportional to $\mathrm{e}^{\gamma t}$; find γ.

iv. (1 pt) Let us now substitute the O-tube with a spherical shell of radius R filled with these two liquids, each of which occupies a hemispherical region inside the shell. In order to keep the interface between the liquids flat, a massless thin rigid circular membrane of radius R is placed in between the liquids; the membrane can rotate frictionlessly inside the sphere, but cannot be bent. Find the instability growth rate γ (defined above) if the heavier liquid occupies the upper half of the sphere.
Hint: the center of mass of a solid homogeneous hemisphere of radius R is at the distance $\frac{3}{8}R$ from the sphere's centre.
The moment of inertia of a solid sphere of mass is given by $I = \frac{2}{5}MR^2$.
Small angle approximations following $\sin\alpha \approx \tan\alpha \approx \alpha$, $\cos\alpha \approx 1 - \alpha^2/2$ can be used.

Part B. Stabilization due to surface tension *(3 points)*

According to the results obtained above, the Rayleigh-Taylor instability growth rate γ is a decreasing function of the size R of the region where the liquid starts moving. This means that small-scale perturbations of the interface shape grow faster and dominate at the initial stage of the instability. However, at very small scales, surface tension may stabilize the instability.





i. (1 pt) Assume that a big rectangular vessel is divided into two compartments with a thin flat membrane, the top view is shown in the figure (a) above, and a vertical cross-section with the vessel being filled with liquids — in the figure (b). The membrane has a long and narrow slit: its length l is much larger than its width d ($l \gg d$). The upper compartment is filled with a liquid of density ρ_2, and the lower compartment — with a liquid of density $\rho_1 < \rho_2$. Initially, the slit is so narrow that the surface tension σ which characterizes the interface between the two liquids stabilizes the Rayleigh-Taylor instability: the interface remains completely flat and horizontal. A cross-section in $x-y$-plane of this configuration is depicted in the figure (b) above, where y-axis is vertical, and z-axis is parallel to the longer edge of the slit. The width of the slit d is increased slowly up to a certain value $d = d_0$, where instabilities start developing, but the instability growth rate γ remains extremely small. A special design guarantees that the deformations of the interface between the two liquids remains strictly 2-dimensional — there is no dependence on the z-coordinate (this design can include, for instance, thin long rods placed onto the interface between the liquids, parallel to the slit). Sketch the new shape of the interface in $x-y$-intersection when $d = d_0$ and when it has become noticeably deformed due to the Rayleigh-Taylor instability.

ii. (1 pt) Consider the same setup as before, but now there are no restrictions on how the interface can be deformed, i.e. the deformation can include dependence on the z-coordinate. Now, the interface becomes unstable at somewhat smaller slit width $d = d_1$. Sketch the shape of the interface when $d = d_1$ and it has become noticeably deformed due to the Rayleigh-Taylor instability, in two intersections with planes parallel to the $x-y$ plane: one at the distance $l/4$ from one end of the slit, and the other — at the distance $l/4$ from the other end of the slit.

iii. (1 pt) Express d_1 in terms of ρ_1, ρ_2, σ, and g.

Part C. Water waves *(2 points)* While a heavy liquid atop of a light one is unstable, the reverse situation of a light liquid atop of a heavy one is stable, and surface shape perturbations will travel along the surface as waves. A particular case of such waves are represented by waves on the free surface of water when the light liquid (air) has a negligibly small density. If the water is deep (much deeper than the wavelength λ of the waves), the speed of sinusoidal waves depends on the wavelength,

$$v = \sqrt{g\lambda/2\pi}.$$

Therefore, all wave speeds are possible, including those which travel in "resonance" with the boat: the boat will always remain at the same trough or at the same crest of the wave, and will propel water *resonantly*, i.e. always at the same value of the phase of the wave. If there are waves which can move in a resonance with a moving object, the moving object will generate these waves — this phenomenon is known as Cherenkov radiation. Generated waves carry away energy and this results in a *wave drag* acting on the object. The wave drag grows rapidly with speed (proportional to the cube of the speed) and is the main limiting factor for the speed of boats. Determine the speed of the boat shown in the aerophoto below (you may take measurements from the map).