COVER

The 2nd Gulf Physics Olympiad — Theoretical Competition

Riyadh, Saudi Arabia — Sunday, April 2nd 2017

- The examination lasts for 5 hours. There are 3 problems worth in total 30 points. **Please note that the point values of the three theoretical problems are not equal.**
- **You must not open the envelope with the problems before the signal of the beginning of competition.**
- **You are not allowed to leave your working place without permission.** If you need any assistance (broken calculator, need to visit a restroom, etc), please raise your hand until an organizer arrives.
- Use only the front side of the sheets of paper.
- For each problem, there are **dedicated Solution Sheets** (see header for the number and pictogram). Write your solutions onto the appropriate Solution Sheets. For each Problem, the Solution Sheets are numbered; use the sheets according to the enumeration. **Always mark which Problem Part and Question you are dealing with.** Copy the final answers into the appropriate boxes of the **Answer Sheets**. There are also **Draft** papers; use these for writing things which you don't want to be graded. If you have written something that you don't want to be graded onto the Solution Sheets (such as initial and incorrect solutions), cross these out.
- If you need more paper for a certain problem, please raise your hand and tell an organizer the problem number; you are given two Solution sheets (you can do this more than once).
- **You should use as little text as possible:** try to explain your solution mainly with equations, numbers, symbols and diagrams. Though in some places textual explanation may be unavoidable.
- **After the signal signifying the end of examination you must stop writing immediately.** Put all the papers into the envelope at your desk. **You are not allowed to take any sheet of paper out of the room.** If you have finished solving before the final sound signal, please raise your hand.

Problem T1. Main sequence stars *(11 points)*

In all your subsequent calculations you may use the following physical constants and their numerical values.

Stefan-Boltzmann constant $\sigma = 5.670 \times 10^{-8}\,\mathrm{W/(m^2K^4)}$

(Note that σT^4 gives the black body thermal radiation power per unit area at temperature T.)

Boltzmann constant $k_B = 1.38 \times 10^{-23}\,\mathrm{m^2 \cdot kg \cdot s^{-2} \cdot K^{-1}}$.

The rest mass of a proton $m_p = 1.67 \times 10^{-27}\,\mathrm{kg}$.

Rest energy of a proton $m_pc^2 = 938\,\mathrm{MeV}$,

where $1\,\mathrm{MeV} = 1.6 \times 10^{-13}\,\mathrm{J}$.

Rest energy of a helium nucleus $m_{\mathrm{He}}c^2 = 3727\,\mathrm{MeV}$.

Rest energy of an electron and positron $m_ec^2 = 0.5\,\mathrm{MeV}$.

Speed of light $c = 3 \times 10^8\,\mathrm{m/s}$,

Universal gas constant $R_g = 8.31\,\mathrm{J \cdot K^{-1} \cdot mol^{-1}}$

Avogadro's number $N_A = 6.02 \times 10^{23}\,\mathrm{mol^{-1}}$

Part A. Lifetime of Sun *(3 points)*

For this Part, the following values can be also used.

The mass of Sun $M_\odot = 2 \times 10^{30}\,\mathrm{kg}$.

The radius of Sun $R_\odot = 7 \times 10^8\,\mathrm{m}$.

Surface temperature of Sun $T_\odot = 6 \times 10^3\,\mathrm{K}$.

i. (0.7 pts) The Sun emits thermal radiation as a perfectly black body. Determine the total radiation power of the Sun (in watts).

ii. (0.5 pts) The Sun maintains its temperature owing to the fusion reaction, the net effect of which can be written as $4p^+ \to {}^4\mathrm{He}^{2+} + 2e^+ + 2\nu_e$, where p^+ denotes a proton, ${}^4\mathrm{He}^{2+}$ — a helium nucleus, e^+ — a positron, and ν_e — an electron neutrino of negligible rest energy. Show that the energy released by such a fusion of four protons is $W_0 = 24\,\mathrm{MeV}$.

iii. (0.5 pts) Antimatter cannot co-exist with matter: upon meeting, a positron and an electron disappear by producing two photons. How much energy per each fusion of four protons into a helium nucleus must leave Sun (carried away by photons and neutrinos) in order to keep it at a thermal equilibrium?

iv. (1.3 pts) Assuming that only the central part of the Sun (the Sun's nucleus) which makes $\frac{1}{8}$ of the total mass of the Sun is hot enough for fusion reaction to take place, and neglecting the energy carried away by neutrinos, estimate the total lifetime of the Sun. Note that there is no convection in the central parts of the Sun, and therefore the particles inside the Sun's nucleus remain trapped therein. Based on your result, comment on the current age of Sun, $\tau_\odot = 5 \times 10^9\,\mathrm{y}$.

Part B. Mass-luminosity relationship of stars *(4.5 points)*

Inside the nuclei of the so-called main sequence stars (such as our Sun), the fusion reaction takes place in a stable regime: if fluctuations were to increase the reaction rate slightly, the increased thermal output would lead to an increase of the pressure, to a thermal expansion of the fusion plasma, and as a result, to a decrease of the reaction rate. The reaction rate grows very rapidly with temperature and because of that, even if the reaction rates in different stars of different masses may differ considerably, the interior temperatures remain fairly similar. In what follows, you may assume that **the temperature of the nuclei of stars is independent of the stellar mass and equal to**

$$T_c = 1.8 \times 10^6\,\mathrm{K};$$

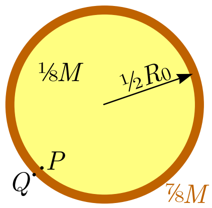
this approximation holds particularly well for stars larger than Sun.

In order to make our next calculations mathematically easier, we make the following additional approximations.

(a) The mass of the stellar core is $\frac{M}{8}$ and its radius is $\frac{R_0}{2}$, where M is the total mass of the star and R_0 — the radius of the star.

(b) The mass density ρ_c, pressure p_c, and temperature T_c inside the stellar core **can be approximately taken to be constant throughout its volume**.

(c) For tasks i–iv, we assume also that all the mass $\frac{7}{8}M$ of the outer layers of the star is concentrated into a very narrow spherical layer of radius $\frac{R_0}{2}$ around the core, see figure. In reality, this is certainly not true — the layer is not narrow. However, this approximation will have only a minor effect on the expression for the pressure (in task iv).



i. (0.4 pts) Express the free fall acceleration immediately above the narrow spherical layer (point Q in figure) in terms of M and R_0.

ii. (0.4 pts) Express the free fall acceleration immediately beneath the narrow spherical layer (point P in figure).

iii. (0.4 pts) Express the gravity force acting on a small piece of the narrow spherical layer in terms of its surface area A, M and R_0.

iv. (0.4 pts) Express the pressure p_c in terms of the radius R_0 and mass M of the star; (we overestimate it only by a factor which is less than two).

v. (1 pt) Derive another expression for the pressure p_c, this time in terms of R_0, M, and the core temperature T_c. Assume that the nucleus of a star is made of a fully ionised hydrogen, i.e. there are free protons and free electrons, **both of which** can be described as an ideal gas.

vi. (0.4 pts) Based on your previous results, express the radius R_0 of a star in terms of its mass M and temperature T_c.

vii. (1.5 pts) The radiative power of a star is limited by at which rate the produced heat can travel through the outer layers of the star and reach the surface. The heat conductivity κ is defined as the proportionality coefficient between the heat flux density (thermal power per unit area) and temperature gradient $\frac{dT}{dr}$, where r is the distance from the centre of the star. For a plasma, the heat conductivity is inversely proportional to its density, $\kappa = f(T)/\rho$. Assume simplifyingly that κ is constant throughout the bulk of a star, up to the near-surface regions at $r = R_0$ where the temperature $T \ll T_c$, and is equal to $\kappa = f(T_c)/\rho_c$. Show that the total radiative power P of a star is proportional to M^γ, and find the exponent γ.

Part C. Proton-proton fusion chain *(3.5 points)*

We say that a constant is fundamental if it cannot be expressed in terms of other fundamental constants; for instance, the Stefan-Boltzmann constant can be expressed in terms of k_B, speed of light c, and Planck's constant $\hbar$. However, majority of the fundamental constants are created artificially by physicists due to a non-fundamental way of choosing the units. For instance, SI system of units needs electrostatic constant k_e , but for Gauss system of units, charge units are such that $k_e = 1$. So, majority of the "fundamental" constants are not really that fundamental, and depend on our (essentially arbitrary) choice of units. However, there are also dimensionless combinations of physical constants, which can be considered as the parameters of our Universe, and which define the way in which matter and fields evolve.

i. (1.5 pts) Find a dimensionless combination α^{-1} and calculate its value using the following subset of fundamental constants (it may happen that only few constants will enter the expression for α^{-1}):

$c = 3 \times 10^8\,\mathrm{m/s}$,
$G = 6.67 \times 10^{-11}\,\mathrm{m^3 \cdot kg^{-1} s^{-2}}$,
$k_B = 1.38 \times 10^{-23}\,\mathrm{J \cdot K^{-1}}$,
$N_A = 6.02 \times 10^{23}\,\mathrm{mol^{-1}}$,
$\hbar = \frac{h}{2\pi} = 1.05 \times 10^{-34}\,\mathrm{J \cdot s}$,
$e = 1.6 \times 10^{-19}\,\mathrm{C}$,
$k_e = \frac{1}{4\pi\varepsilon_0} = 8.99 \times 10^9\,\mathrm{m \cdot F^{-1}}$.

Note that any power of α is also dimensionless; you are asked to find the simplest combination of constants which yields $\alpha^{-1} > 1$.

Hint: before applying dimensional analysis, all units need to be expressed using the base units (m, s, A, K, kg, mol).

ii. (1 pt) The first and limiting step in the fusion of four protons into a helium atom inside a star of sub-solar mass is the fusion of two protons,

$$p^+ + p^+ \rightarrow {}^2\mathrm{H}^+ + e^+ + \nu_e.$$

This process is obstructed, however, by a coulomb repulsion of two protons. You may assume that until the distance between the centres of two protons remains larger than the proton radius $r_p = 0.85 \times 10^{-15}\,\mathrm{m}$, there is only a Coulomb force; at distances smaller than r_p, an attractive strong force steps into play and dominates over the Coulomb force. Estimate the temperature T' required for the fusion of two protons if there were no quantum-mechanical effects. Compare this result with the value of $T_c \approx 1.8 \times 10^6\,\mathrm{K}$.

iii. (1 pt) What enables the fusion of stellar hydrogen is the quantum-mechanical tunnel effect. With this task, you'll learn that the fusion reaction rate depends on the dimensionless parameter α, thus we can say that the parameter α defines the production rate of heavier nuclei in our Universe. (It appears that in a slightly different Universe with a slightly different value of α, no carbon nuclei neccessary for the existance of life would have been produced[1].)

It appears that a particle can tunnel through an energy barrier (a region in space where the potential energy $\Pi(r)$ is larger than the total energy W) with probability

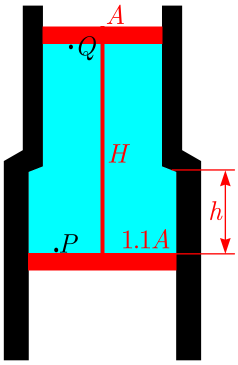
$$p \approx \exp\{-2\hbar^{-1} \int \sqrt{2m[\Pi(r) - W]}\mathrm{d}r\},$$

where the integral is to be taken over the range at which $\Pi(r) > W$. Express the tunnelling probability for the proton-proton fusion reaction for head-on collision of two counter-moving protons of speed v in terms of α, v and c. You may assume that the proton radius r_p is much smaller than the radius $r_\star$ at which the proton "dives into the tunnel" $[\Pi(r_\star) = W]$, and make use of the equality $\int_0^a \sqrt{\frac{1}{x} - \frac{1}{a}}\mathrm{d}x = \frac{\pi}{2}\sqrt{a}$.

[1] J. Barrow and F. Tipler, *The Anthropic Cosmological Principle*, Oxford, (1988)

Problem T2. Water tube *(8 points)*

Consider a tube which is obtained when two metallic cylinders are welded together as shown in the figure. Upper cylinder has internal cross-sectional area $A = 10\,\mathrm{cm}^2$, and the lower one — $1.1A = 11\,\mathrm{cm}^2$. Two pistons are connected with a narrow (rigid but light) steel bar of length $H = 30\,\mathrm{cm}$; the distance from the lower piston to the welding area is $h = 10\,\mathrm{cm}$. The space between the pistons is filled with water of density $\rho = 1000\,\mathrm{kg/m}^3$ and temperature $T = 20\,°\mathrm{C}$. The mass of each of the pistons $\frac{m}{2} = 50\,\mathrm{g}$ (neglect the mass of the rod connecting them) free fall acceleration $g \approx 10\,\mathrm{m/s}^2$ and the atmospheric pressure $p_0 = 1 \times 10^5\,\mathrm{Pa}$. The tube stands vertically on a solid horizontal surface; the pistons can move freely up and down, friction force can be neglected. The distance between the bottom of the lower piston and the horizontal surface is more than $20\,\mathrm{cm}$.



i. (0.5 pts) Let p_P denote the pressure at a point P at the bottom of the water column, and p_Q — at a point Q at the bottom. Find $p_P - p_Q$.

ii. (1.5 pts) Consider the two pistons, steel bar, and water column as a single compound body. Make a sketch and mark on it all the forces acting on this compound body by arrows (denote them by letters — $\vec{F}_1$, $\vec{F}_2$, etc.). Determine the values of all these forces.

iii. (1.2 pts) Determine the values of p_P and p_Q.

iv. (0.8 pts) Determine the tension force T in the steel bar.

v. (1 pt) Now, the whole system is slowly raised to a height $L = 25\,\mathrm{cm}$ (this is the distance between the horizontal surface and the bottom edge of the tube), and released. The system falls due to gravity, hits the surface (assume the impact to be plastic, i.e. the kinetic energy of the metallic tube is converted into heat), remains standing vertically on the horizontal surface for a brief period of time τ, and jumps up into air. Why does it jump? Provide a qualitative explanation.

vi. (3 pts) Find the duration τ during which the tube remains standing on the surface (after falling and before jumping).

Problem T3. Accelerating shock wave *(11 points)*

In interstellar space, shock waves can accelerate charged particles to very high energies. We shall use an idealized model of a shock wave, and assume that it is a potential barrier of a constant height $-V_0$ which moves with a constant velocity w along the x-axis:

$$V(x,y,z,t) = -V_0 \qquad \text{if } x < wt;$$
$$V(x,y,z,t) = 0 \qquad \text{if } x > wt.$$

In the frame where the shock wave is at rest, the energy of an electron is conserved. This means that as long as the kinetic energy of an electron of mass m and charge $-e$ moving towards the shock wave is insufficient ($\frac{1}{2}mu^2 < eV_0$, where u denotes the speed with which the electron is approaching the shock wave), it is reflected back from the shock wave **in the same way as an elastic ball bounces from a rigid wall. In what follows, unless otherwise mentioned, we assume that the electron is bounced elastically by the shock wave**. You can always use the parameters e, V_0, m, B, and w to express your answers. Unless otherwise specified, the velocity of the electron is assumed to be non-relativistic.

i. (1 pt) Let the initial speed of the electron be $\vec{v} = (v_x, v_y, v_z)$, with $v_x < w$. Determine the velocity $\vec{v}'$ (i.e. the components v'_x, v'_y, v'_z) of the electron after being hit by the shock wave.

ii. (1 pt) Now, there is also an homogeneous magnetic field of induction B, parallel to the z-axis. At the beginning, electron rests at the origin, and at $t = 0$ is hit by the shock wave. Sketch qualitatively the trajectory drawn by the electron; cover the time period from $t = 0$ until at least $t = \frac{\pi m}{Be}$.

iii. (0.5 pts) Find the curvature radius of the electron's trajectory immediately after its first collision with the shock wave.

iv. (1 pt) The electron undergoes soon, at $t = t_2$, a second impact; write down an equation for determining t_2. Use numerical calculation to obtain an expression for t_2.

v. (0.5 pts) Determine the average x-directional velocity v_x of the electron (averaged over the time interval τ between two subsequent collisions of the electron with the shock wave).

vi. (1.5 pts) As time goes on, the electron undergoes many collisions with the shock wave. Show that during its motion, $v_y + kx = \text{const}$, where k is a constant; express k in terms of e, m and B.

vii. (1 pt) From now on, let us consider the limit $t \gg \frac{2\pi m}{Be}$. Determine the average y-directional acceleration a_y of the electron (express it in terms of e, m and B or constant k introduced by task vi).

viii. (1 pt) It appears that at the limit $t \gg \frac{2\pi m}{Be}$, the time interval τ between subsequent collisions becomes shorter and shorter, hence we can assume that $\tau \ll \frac{2\pi m}{Be}$. This means that during a time interval between two subsequent collisions the velocity vector of the electron will change only by a very small angle and hence, its acceleration vector $\vec{a} = (a_x, a_y)$ can be assumed to be constant.

Let us use now *the shock wave's frame of reference*, and consider the electron's *phase diagram*, i.e. a diagram which describes the state of the electron as a point in the $x' - p'_x$-plane, where the vertical axis $p'_x = m(v_x - w)$ corresponds to the x'-component of the momentum, and $x' = \int (v_x - w)\mathrm{d}t$ denotes the distance from the shock wave. Depict qualitatively the electron's *phase trajectory*, i.e. the curve drawn in phase diagram during one period (between two subsequent collisions of the electron with the shock wave). Grades for this task are based purely on the shape of the curve.

ix. (1.5 pts) As time goes on, the width and height of the phase trajectory will change; however, it appears that the surface area of the region surrounded by the phase trajectory (referred to as the *adiabatic invariant*) will remain constant with a very good precision. For an initially resting electron, the adiabatic invariant appears to be approximately equal to $\frac{1.36(mw)^2}{Be}$. Determine the total kinetic energy W_f of the electron when it falls behind the shock wave; express it in terms of e, V_0, and ε, which is defined as $\varepsilon \equiv \frac{2eV_0}{mw^2}$; assume that $\varepsilon \gg 1$.

x. (2 pts) *This final task is independent from the previous tasks.* Consider the propagation of a shock wave as described before, but under the absence of a magnetic field. A relativistic electron moves parallel to the front (in the laboratory frame, the perpendicular component of its velocity is strictly zero). Assuming that $mw^2 < eV_0$ and $w \ll c$ (with c denoting the speed of light), what should be the relativistic energy of the electron so that it could fall behind the shock wave? You can use any reasonable approximations.