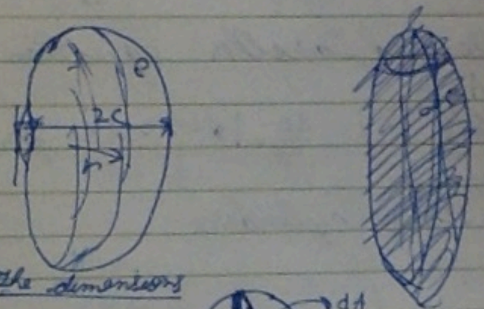
We model the disk (radius r) as ~~an~~ a spheroid, with radius r, and 'thickness' 2c, c→0, so that the field produced by the spheroid may be identical to that produced by a disk, with its ~~su~~ charge density ~~projected onto the~~ same as ρ (volume charge density) projected onto the ~~su~~ circular ~~surfa~~ plane passing through its centre

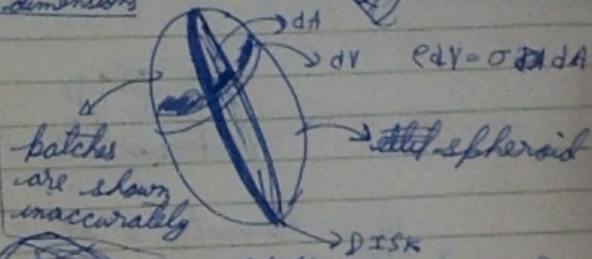


(i) The dimensions

$c \to 0$

⇒ Very thin

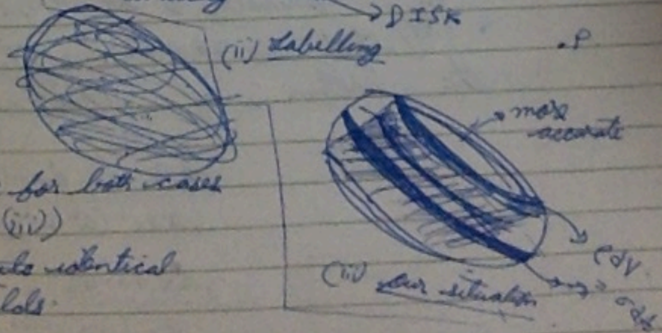
⇒ spheroid → 0-thickness disk



(ii) Labelling

⇒ Distances of points from P, are same for both cases (figure (iii))

⇒ ~~Same~~ identical fields



(iii) Our situation

As we'd found earlier, we needed to neutralise a spring-type force, which on

According to the given theorem, along with symmetry,

~~If~~ $\phi(x,y,z) = A(x^2+y^2)+Bz^2$,

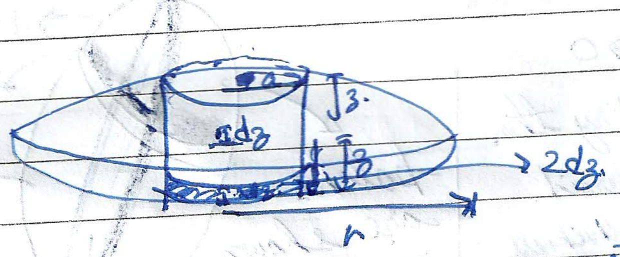
the 'circle' being in the x,y plane and the centre at the origin.

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0}.$$

$$\Rightarrow \quad 4A+2B = -\frac{\rho}{\varepsilon_0} \quad \text{—(i)}.$$

We need another equation to determine A & B.

Let's $\int \frac{dE}{dz}\Big|_z (0,0,0)$. for infinitesimally thin cylinders



$\frac{dE}{dz}$ Symmetry $\Rightarrow$ at $(0,0,dz)$, net fields same as that ~~$2\pi a\, da$~~ due to a 'ring' at the bottom, of thickness $2dz$.

$$\therefore d^2E = \frac{2\pi a\rho \cdot da \cdot dz}{4\pi\varepsilon_0 \cdot (a^2+z^2)} \cdot \frac{z}{(a^2+z^2)^{1/2}}$$

$$= \frac{z a \rho\, da \cdot dz}{\varepsilon_0 (a^2+z^2)^{3/2}}$$

$$\Rightarrow d\left(\frac{dE}{dz}\right) = \frac{a z \rho\, da}{\varepsilon_0 \cdot (a^2+z^2)^{3/2}}.$$

$$\Rightarrow \frac{dE}{dz} = \frac{e}{\varepsilon_0}\int_0^r \frac{a z\, da}{(a^2+z^2)^{3/2}}$$

$$\frac{a^2}{r^2} + \frac{z^2}{c^2} = 1$$

$$\cancel{\Rightarrow z = \pm r\sqrt{1-\frac{z^2}{c^2}}} \qquad \Rightarrow z^2 = c^2\left(1-\frac{a^2}{r^2}\right)$$

$$\therefore \frac{dE}{dz} = \frac{e}{\varepsilon_0}\int_0^r \frac{a c\sqrt{1-\frac{a^2}{r^2}}\ .da}{\left(a^2 + c^2\left(1-\frac{a^2}{r^2}\right)\right)^{3/2}}$$

$$= \frac{e}{2\varepsilon_0}\int_0^r \frac{c\sqrt{1-\frac{a^2}{r^2}}\ da^2}{\left(a^2\left(1-\frac{c^2}{r^2}\right)+c^2\right)^{3/2}}$$

$$= \frac{e}{2\varepsilon_0}\int_0^r \frac{c r^2\sqrt{1-\left(\frac{a}{r}\right)^2}\ d\left[\left(\frac{a}{r}\right)^2\right]}{r^3\left(\frac{a^2}{r^2}\left(1-\frac{c^2}{r^2}\right)+\frac{c^2}{r^2}\right)^{3/2}}$$

$$= \frac{e}{2\varepsilon_0}\int_0^r \frac{c}{r}\ \frac{\sqrt{1-\left(\frac{a}{r}\right)^2}\ d\left(\left(\frac{a}{r}\right)^2\right)}{\left(\left(\frac{a}{r}\right)^2\left(1-\left(\frac{c}{r}\right)^2\right)+\left(\frac{c}{r}\right)^2\right)^{3/2}}$$

$$\gamma \equiv \frac{c}{r}.$$

$$p \equiv a^2/r^2.$$

$$\therefore \frac{dE}{dz} = \frac{e}{2\varepsilon_0}\int_0^1 \frac{\gamma\sqrt{1-p}\ dp}{\left(p(1-\gamma^2)+\gamma^2\right)^{3/2}}.$$

$$= \frac{e\,\gamma}{2\varepsilon_0(1-\gamma^2)^{3/2}}\int_0^1 \frac{\sqrt{1-p}\ dp}{\left(p+\frac{\gamma^2}{1-\gamma^2}\right)^{3/2}}.$$

$$= \frac{e}{2\varepsilon_0}\cdot\frac{\gamma}{(1-\gamma^2)^{3/2}}\cdot\int_0^1 \frac{\sqrt{1-p}\ dp}{\left(p+\frac{\gamma^2}{1-\gamma^2}\right)^{3/2}}.$$

From Wolfram Alpha, (integration), after expanding the expression into a Taylor series in γ upto $\Theta(\gamma)$.

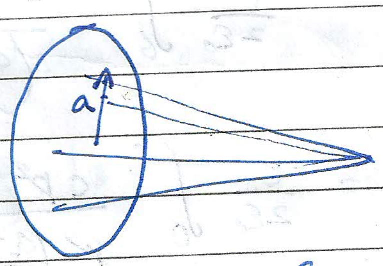
$$\frac{\rho}{2\varepsilon_0} \frac{\cancel{\pi}/2}{(1-\gamma^2)^{3/2}} (2 - \pi\gamma) = -2B \left[-\frac{\partial E_z}{\partial z} - \frac{\partial^2 E}{\partial z^2} \right]$$

$$\Rightarrow \frac{\rho}{\varepsilon_0}\left(1 - \frac{\pi}{2}\gamma\right) = -2B.$$

$$\Rightarrow 2B = -\frac{\rho}{\varepsilon_0} + \frac{\pi}{2}\frac{\rho}{\varepsilon_0}\gamma = -\frac{\rho}{\varepsilon_0} - 4A \quad [\text{from (i)}]$$

$$\Rightarrow A = -\frac{\pi\rho}{8\varepsilon_0}\gamma$$

. . . .



As in my earlier solution (the correct one), $E_{radial} = \frac{q}{4\pi\varepsilon_0 L^3} a$.

$q > 0 \Rightarrow \rho < 0$, and vice versa, to 0-out this field.

$\because$ we have a spring z-type field;

$$E_{radial} = \frac{-q}{4\pi\varepsilon_0 L^3} a,$$

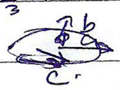
$$V_{reqd.}(x, y, 0) = \frac{q}{8\pi\varepsilon_0 L^3}(x^2+y^2) \Rightarrow A = \frac{q}{8\pi\varepsilon_0 L^3}$$

When the $V_{ellipsoid}(x, y, 0) = V_{reqd.}(x, y, 0)$ were satisfied.

Hence, $-\frac{\pi\rho}{8\varepsilon_0}\frac{c}{r} = \frac{q}{8\pi\varepsilon_0 L^3}$.

$$\Rightarrow \rho c = \frac{-qr}{\pi^2 L^3}$$

$$\Rightarrow Q_{net} = \frac{4}{3}\pi \rho r^2 c = -\frac{4}{3}\pi r^3 \cdot \frac{qr}{\pi^2 L^3}$$ [Volume of ellipsoid $= \frac{4}{3}\pi abc$



]

$$= -\frac{4}{3}\frac{q r^3}{\pi L^3}.$$

$\Rightarrow Q_{left}$ (~~to~~ be distributed on the ~~eq~~ non-equipotential disk)

$$= \frac{4}{3}\frac{q r^3}{\pi L^3}.$$

$\sigma_{(a)}$ $= 2\rho z(a)$. (projection)

$$= 2\rho c\sqrt{1-\frac{a^2}{r^2}} = -\frac{2qr}{\pi^2 L^3}\sqrt{1-\frac{a^2}{r^2}}.$$

From the theorem (claim I) in my first solution (wrong solution, but correct claim),

Q_{left} gives $\sigma_{left} = \frac{4}{3}\frac{q r^2}{\pi^2 L^3}\frac{1}{2\pi r}\frac{1}{\sqrt{r^2-a^2}}$

$$= \frac{2}{3}\frac{q r}{\pi^2 L^3}\cdot\frac{1}{\sqrt{1-\frac{a^2}{r^2}}}.$$

$$\therefore \sigma_{net}(a) = \frac{2qr}{\pi^2 L^3}\left(\frac{1}{3}\frac{1}{\sqrt{1-\frac{a^2}{r^2}}} - \sqrt{1-\frac{a^2}{r^2}}\right)$$

~~As a check, e.g.~~

$$F = \frac{qL}{2\varepsilon_0}\int_0^r \frac{a\,\sigma(a)\cdot da}{(L^2+a^2)^{3/2}}.$$

~~Performing expanding the integrand into a Taylor series upto (a/L²) and~~

Integrating on A CAS, $|\vec{F}_{interaction}| = \boxed{\frac{2}{15\pi^2}\frac{q^2 r^5}{\varepsilon_0 L^7}}$

attractive (−ve).

P.S (i) The integrand in the final step ~~evaluated to $\frac{4}{45}$~~ was expanded to

$$\int \left[-\frac{3}{2} \frac{a^3}{L^5} \sigma(a) \right] da.$$

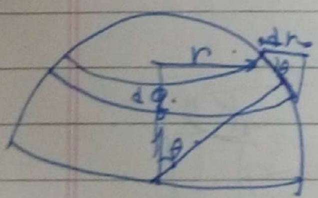
$$\left[\int a\sigma(a)\, da = 0 \Rightarrow \text{next order in } \left(\frac{a}{L}\right) \right.$$
$$\left. \because a \ll L \right]$$

(ii) A proof of Claim I and some of the other things proved in previous solutions is attached.

Proof of Claim I

If the charge Q on the disk were distributed uniformly on a spherical shell 'circumscribing' the disk, the disk would be an equipotential (0 field inside uniformly charged spherical shell).

So we project $2\sigma_{sphere} = \frac{Q}{2\pi R^2}$ on the sphere onto the disk.



[$2\pi R^2$, ∵ we've been talking, all this while of a single surface. Now, Q distributes on 2 surfaces, ∴ $\frac{Q}{4\pi R^2} \times 2 = \frac{Q}{2\pi R^2}$]

$$dq = 2\sigma_{sphere}\, 2\pi r\, dr \sec\theta = \frac{2 \times 2\pi r\, dr\, \sigma_{sphere} R}{\sqrt{R^2 - r^2}}$$

$$= \sigma_{disk}\, 2\pi r\, dr.$$

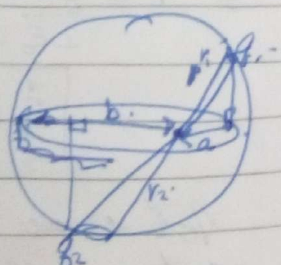
$$\Rightarrow \sigma_{disk} = \frac{Q}{2\pi R \sqrt{R^2 - r^2}}$$

face.

The projection works, ∵,

$\frac{r_1}{a} = \frac{r_2}{b}$ (Similar Δ's).

$\Rightarrow$ just as $\frac{q_1}{r_1^2} = \frac{q_2}{r_2^2}$, $\frac{q_1}{a^2} = \frac{q_2}{b^2}$



(Cancellation in pairs of patches, akin to proof of $E = 0$ inside a sphere)*

Q.E.D.