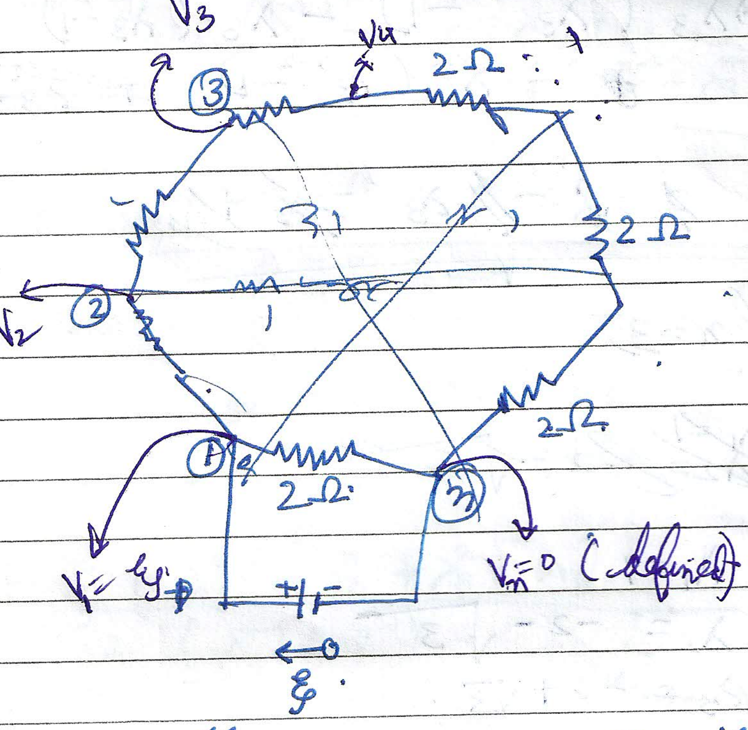
2



Presently, we remove the units for clarity.*
Number the vertices, as shown above. (encircled).
Let the potential of the k^{th} vertex be V_k.
Then, ~~current~~ charge-conservation gives

$$\sum_{\substack{\text{all edges}\\ \text{(wires)}}} I_i = 0,$$

i.e.,

$$\frac{V_{k-1}-V_k}{2} + \frac{V_{k+1}-V_k}{2} + (V_1-V_k) + (V_2-V_k) + \dots + (V_{k-2}-V_k) + (V_{k+2}-V_k) + \dots + (V_n-V_k) = 0$$

as long as $k \neq 1, n$. [Current flow from battery]

Hence, simplifying,

(i) — $$\frac{1}{2}V_{k-1} + \frac{1}{2}V_{k+1} - (n-2)V_k + V_{k-2} + \left(V_1+V_2+\dots+V_{k-3}+V_{k+2}+\dots+V_n\right) = 0$$

$k \to k-1$

(ii) — $$\Rightarrow \frac{1}{2}V_{k-2} + \frac{1}{2}V_k - (n-2)V_{k-1} + V_{k+1} + \left(V_1+V_2+\dots+V_{k-3}+V_{k+2}+\dots+V_n\right) = 0$$

* We'll put back the units in the final answer.

The 2 equations ((i) and (ii)) hence obtained hold for $k \neq 1, n$ & $k-1 \neq 1, n$, i.e, $k \neq 1, 2, n$.
[$k < n+1$, anyway].

Subtracting one equation from the other, we obtain a linear homogeneous recurrence in V_k's.:

(iii)$_{k-2}$ … $V_{k+1} = (2n-3)(V_{k-1} - V_k) + V_{k-2}$; $k \neq 1, 2, n$.

This yields V_{k+1}'s from V_{k-1}, V_k and V_{k-2};

it's easy to see that there're $(n-3)$ such equation. Defining $V_n \equiv 0$, $V_1 = \xi$; so we have $(n-2)$ variables. A symmetry condition works as the final equation, as we shall see.

The condition is,

$$(V_1 - V_2) = (V_{n-1} - V_n),$$

(iv) … i.e, $V_2 + V_{n-1} = \xi$.

When we solve the (iii)$_{k-2}$ for $V_n = V(n)$,

we may put (use our equation for) $n = 4, 5, \dots, n$, i.e, all values we get directly from the equation (of V_k's).
In other words, range (n) = range $(k+1)$.
$= \{ m \mid m \in \mathbb{Z}, m \in [4, n] \}$.

Our initial conditions would be (iv), $V_n = 0$ and $V_1 = \xi$. — [(v)&(vi)]

Now, let's get the eigenvalues of (iii).
Putting $V_k = \lambda^k$,

$$(\lambda^{k+1} - \lambda^{k-2}) = (2n-3)(\lambda^{k-1} - \lambda^k)$$

$$\Rightarrow \lambda^3 + (2n-3)(\lambda^2 - \lambda) - 1 = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2+\lambda+1) + (2n-3)\lambda\cdot(\lambda-1) = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2 + (2n-2)\lambda+1) = 0$$

$$\Rightarrow \lambda = 1,\ 1-n-\sqrt{n^2-2n},\ 1-n+\sqrt{n^2-2n}.$$

We define $\lambda_1 \equiv 1$,

$$\lambda_2 \equiv 1-n+\sqrt{n^2-2n}$$

and $\lambda_3 \equiv 1-n-\sqrt{n^2-2n}$.

$V_n = A_1\lambda_1^n + A_2\lambda_2^n + A_3\lambda_3^n$; where A_1, A_2 and A_3 must be determined from the initial conditions.

Our initial conditions ~~are~~ yield:

$$\begin{cases} A_1 + A_2\lambda_2^n + A_3\lambda_3^n = 0. \\ A_1 + A_2\lambda_2 + A_3\lambda_3 = \xi \\ (A_1 + A_2\lambda_2^{n-1} + A_3\lambda_3^{n-1}) + (A_1 + A_2\lambda_2^2 + A_3\lambda_3^2) = \xi \end{cases}$$

Using a CAS, we get, (we even use $\lambda_2\lambda_3 = 1$ to simplify)

$$A_1 = \frac{\xi}{2},$$

$$A_2 = \frac{\lambda_3^n + \lambda_3}{2(\lambda_3^{n-1} - \lambda_2^{n-1})}\xi$$

and $$A_3 = \frac{\lambda_2^n + \lambda_2}{2(\lambda_2^{n-1} - \lambda_3^{n-1})}\xi.$$

Now for $R_{eq} \equiv R$.

$R_{eq} = \dfrac{\xi}{I_1}$, I_1 being the total current flowing out of vertex 1.

$$\therefore R = \frac{\mathcal{E}}{\sum_i \frac{(V_1 - V_i)}{R_{1i}}}$$

$$= \frac{\mathcal{E}}{\frac{(V_1 - V_n)}{2} + \frac{V_1 - V_2}{2} + \sum_{i \neq n,1} (V_1 - V_i)}$$

$$= \frac{\mathcal{E}}{\cancel{V_1 - \frac{V_n}{2}}\,(n-1) V_1 - \frac{V_n}{2} - \frac{V_2}{2} - (V_2 + V_3 + \cdots + V_{n-1})}$$

$$= \frac{1}{\left(\frac{n}{2} - \frac{3}{4}\right) - \frac{A_2 \lambda_2^2}{2} - \frac{A_3 \lambda_3^2}{2} - A_2 \lambda_2^3 \frac{(\lambda_2^{n-3} - 1)}{\lambda_2 - 1} - A_3 \lambda_3^3 \frac{(\lambda_3^{n-3} - 1)}{\lambda_3 - 1}} \;\Omega$$

[Substituting values, and summing $\Sigma \lambda_2^k, \Sigma \lambda_3^k$, $k = 2, 3, \ldots, n-1$]

(Note that we've put $A_1 = \frac{\mathcal{E}}{2}$.

Also, we've redefined $A_2 \equiv \frac{\text{old} A_2}{\mathcal{E}}$,

$A_3 \equiv \frac{\text{old} A_3}{\mathcal{E}} = \frac{\lambda_2^n + \lambda_2}{2(\lambda_2^{n-1} - \lambda_3^{n-1})}$ (similarly for A_2))

Typing $n=3$ ~~into~~ in the CAS gives $R = \frac{4}{3}\,\Omega$.

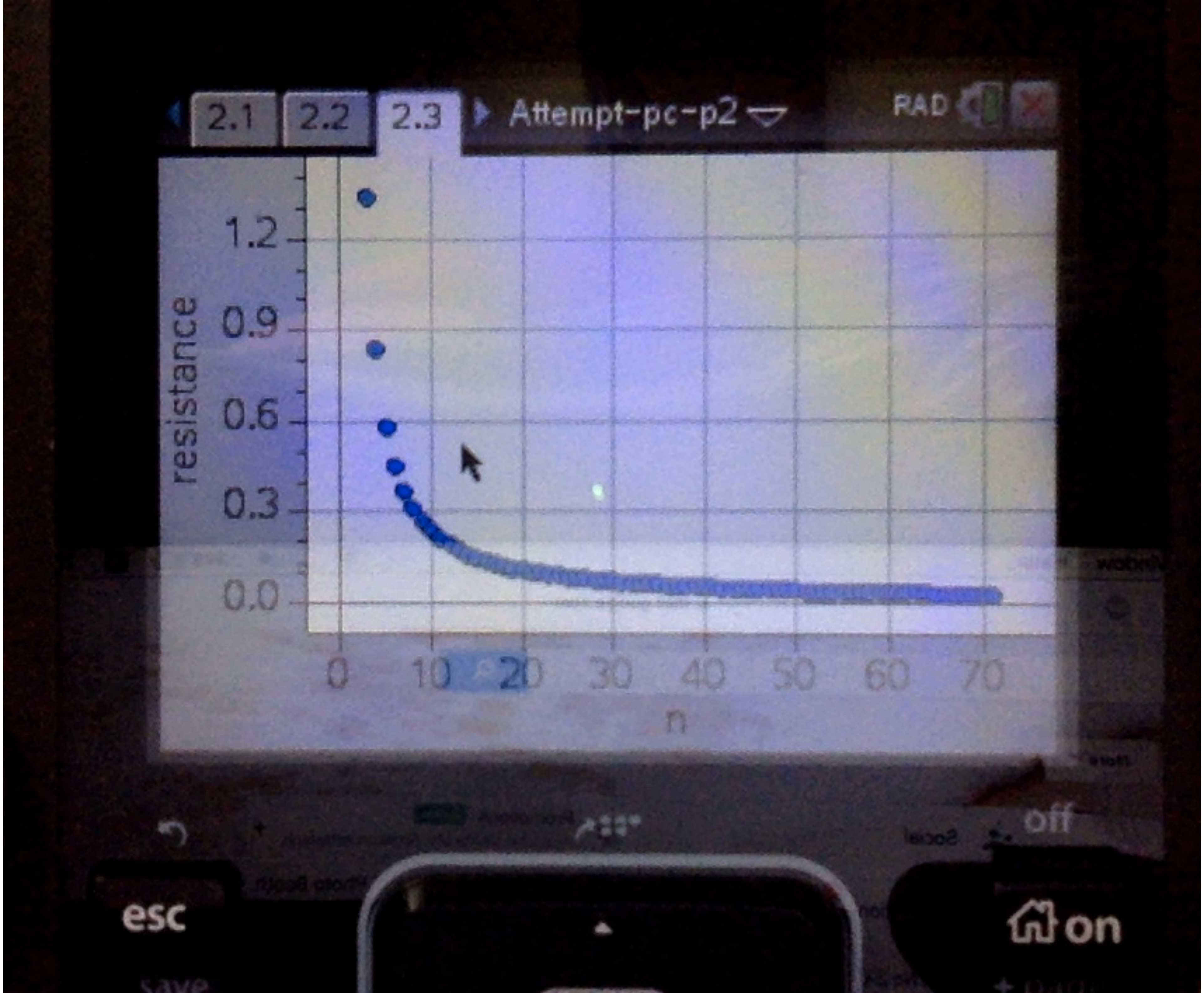
$n=4$ gives $R = \frac{5}{6}\,\Omega$

$n=5$ gives $R = \frac{32}{55}\,\Omega$.

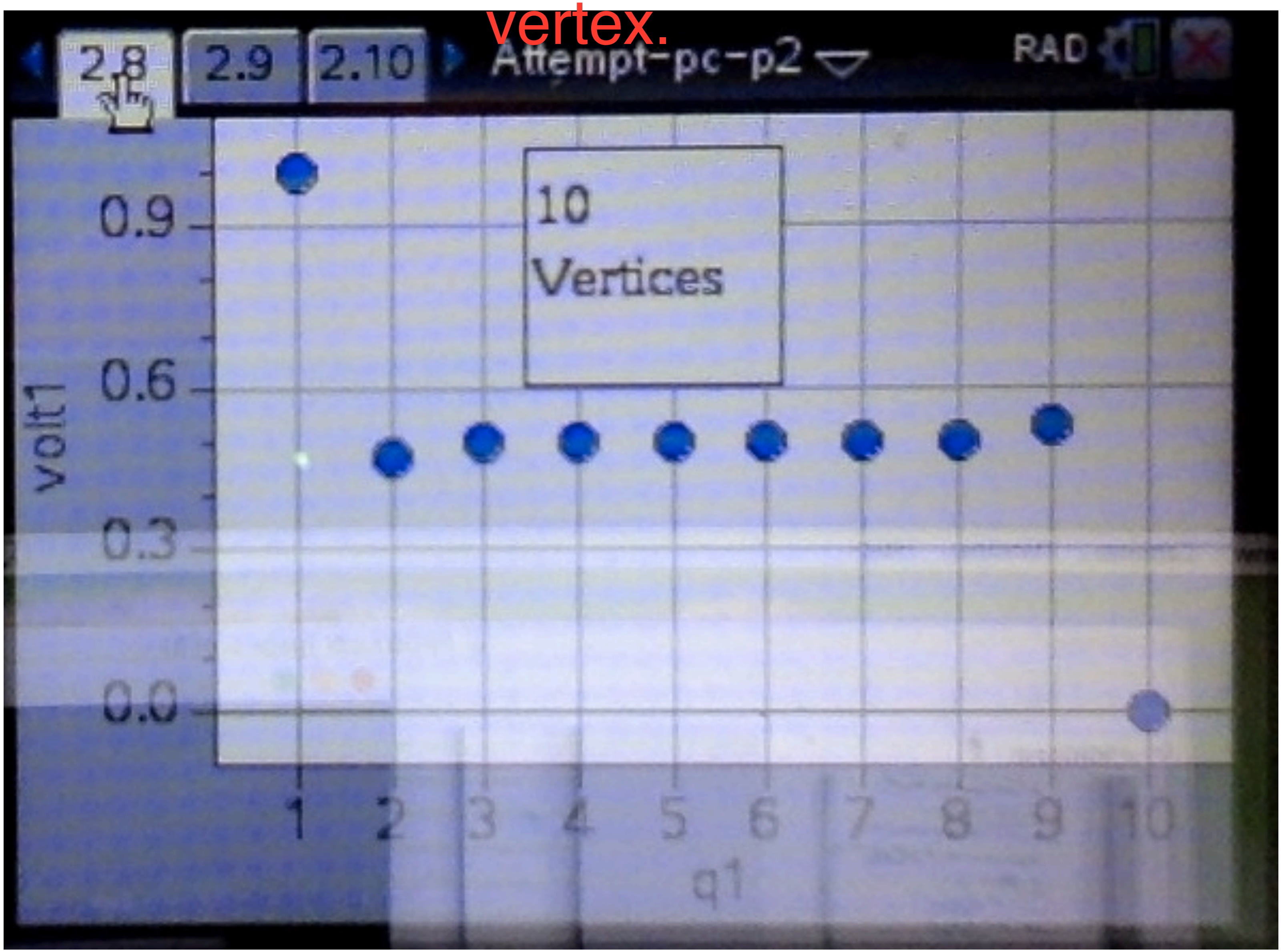
P.S.: (i) The solutions (for A_1, A_2, A_3) may easily be verified by substitution.

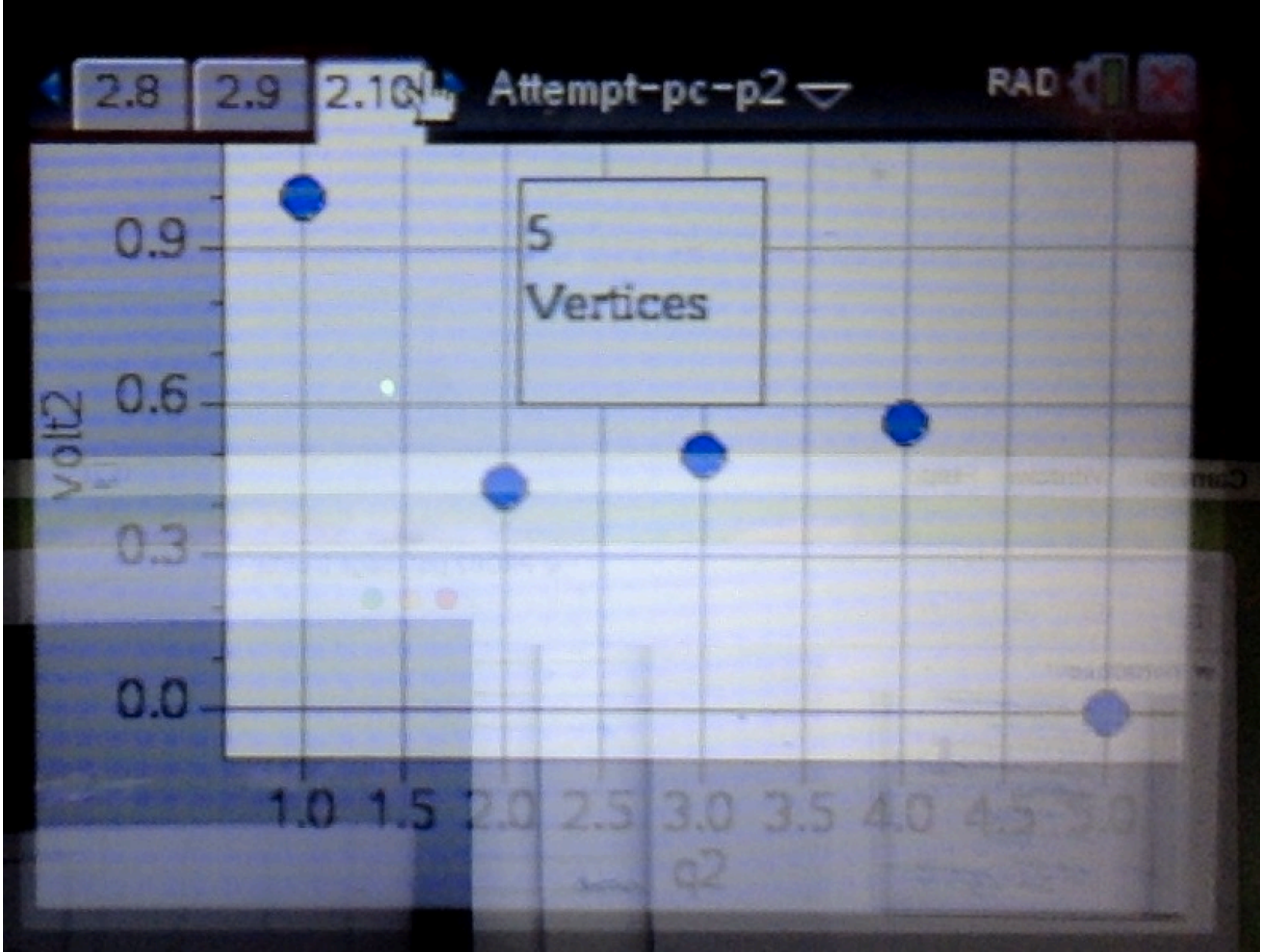
(ii) $\lambda_2 \lambda_3 = \left(1 - n + \sqrt{n^2 - 2n}\right)\left(1 - n - \sqrt{n^2 - 2n}\right) = 1$, used while simplifying the CAS' results. (product of roots of $(\lambda^2 + (2n-2)\lambda + 1) = 0$)

(iii) We provided no proof for linear independence of eq (IV) from other equations. We got an answer, however, so we know that ~~the~~ it was linearly independent.

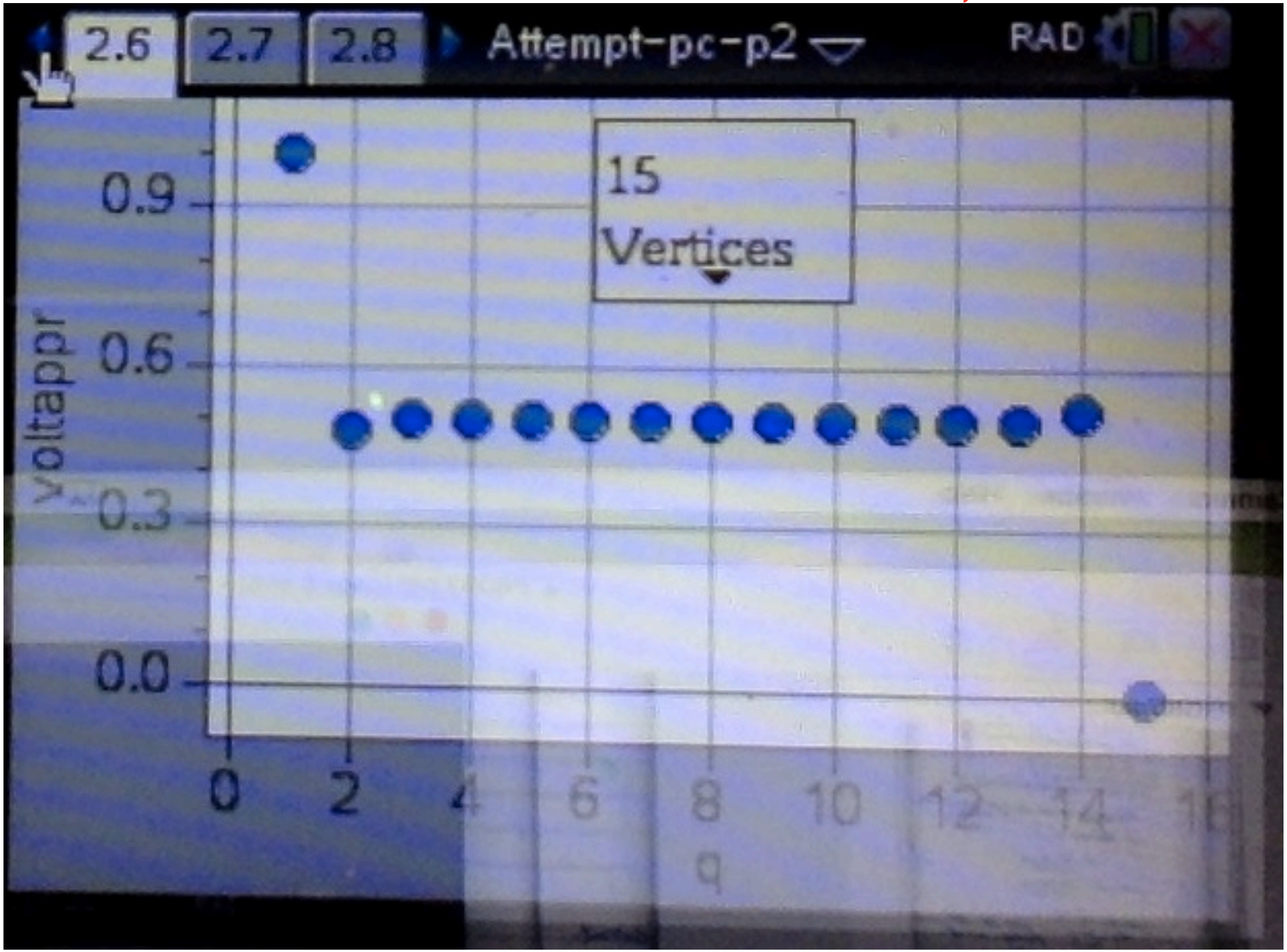

2.1
2.2
2.3
Attempt-pc-p2
RAD
resistance
1.2
0.9
0.6
0.3
0.0
0
10
20
30
40
50
60
70
n
esc
off
on

Some rather not-so-intuitive oscillations of potentials occur, as we move from vertex to vertex.

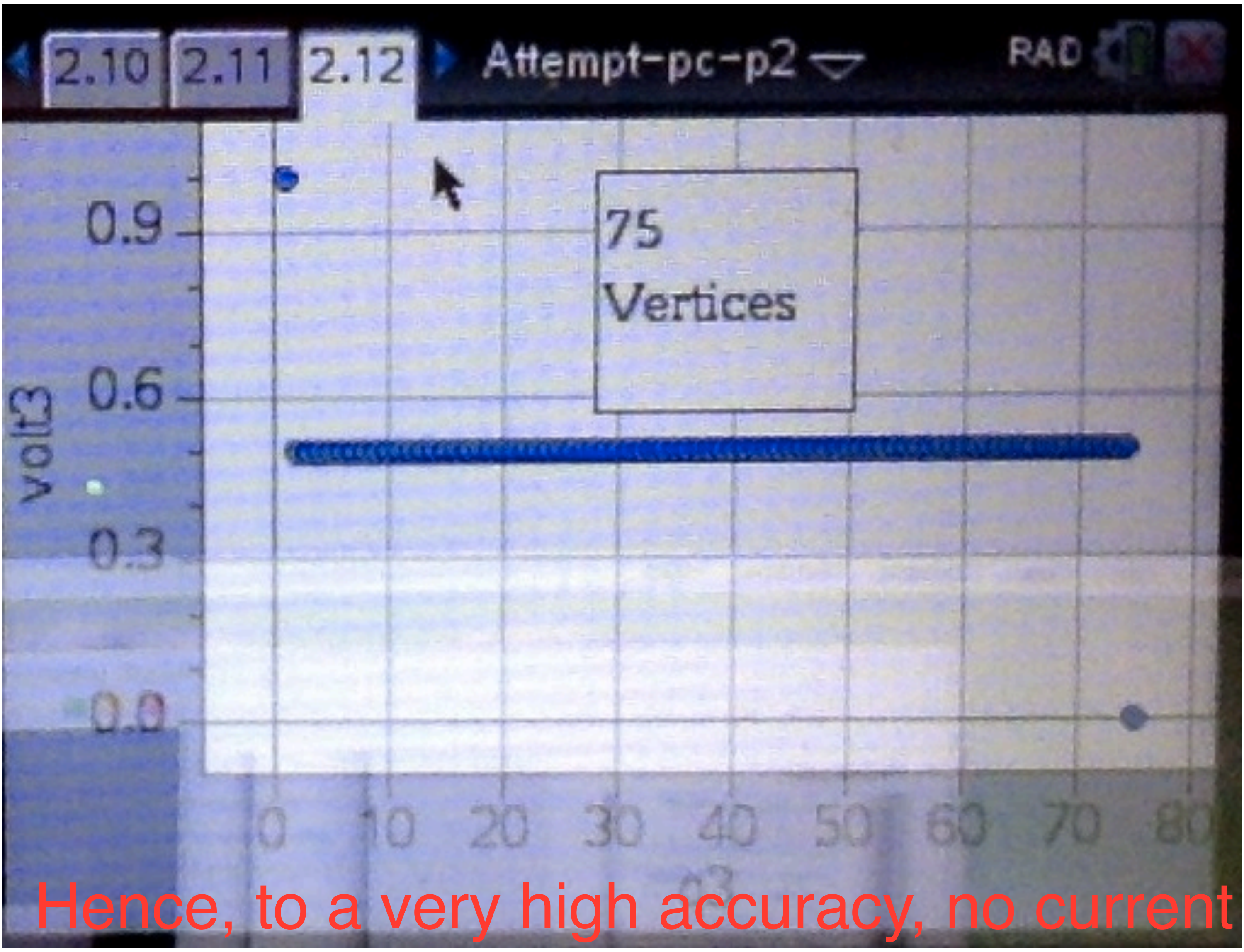



2.8
2.9
2.10
Attempt-pc-p2
RAD
5
Vertices
volt2
0.9
0.6
0.3
0.0
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
q2

TAmplitudes seem to decrease on increasing the total number of vertices, n.



Potentials not near the the endpoints are very close to 0.5 ξ , and are ξ and 0 at the endpoints.It’s surprising how fast the convergence to 0.5 ξ occurs with increase in n and the vertex number.



Hence, to a very high accuracy, no current flows through any wire not connected to a battery terminal, even for moderately large n !!! Each vertex receives current, mainly from the battery, and loses it, mainly to the battery,almost as if it were a parallel connection of resistors!!!

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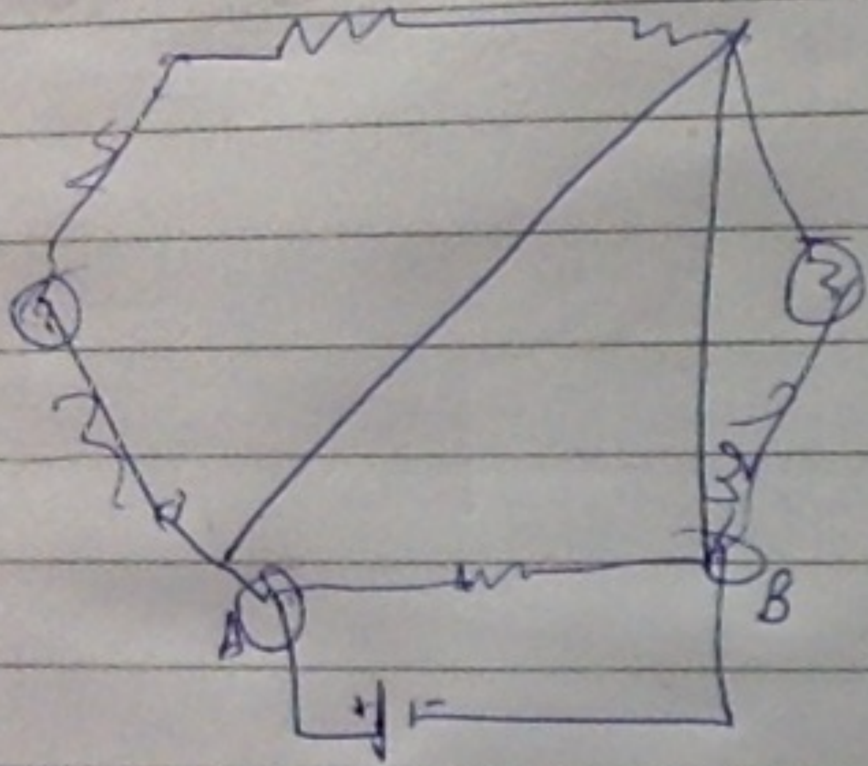


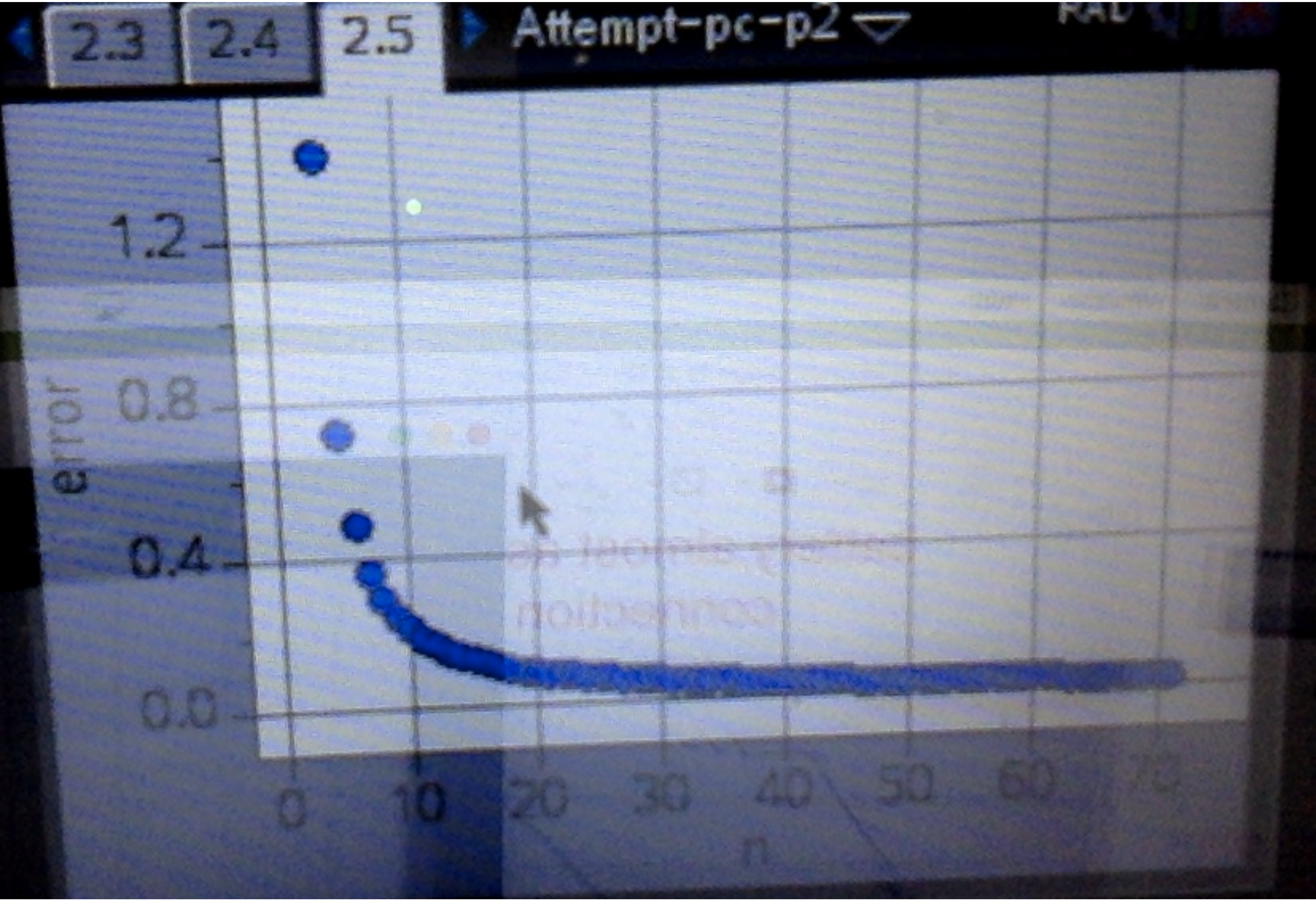
$(n-4)$ 2Ω ~~as~~ end-to-end connections -

1- 2Ω (AB).

$\therefore R_{eq} \approx \frac{2}{n-3}\,\Omega$ for large n.

R_{eq} vs n, does, indeed resemble a hyperbola. $\left(\frac{2}{n-3}\right)$

The next page is a graph of the error in the $R_{eq} \approx \frac{2}{n-3}\,\Omega$ approximation vs n.



2.3
2.4
2.5
Attempt-pc-p2
RAD
error
1.2
0.8
0.4
0.0
0
10
20
30
40
50
60
70
n