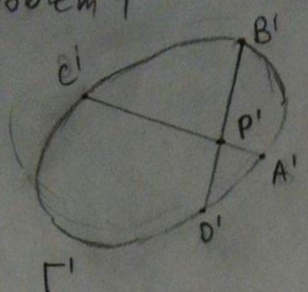
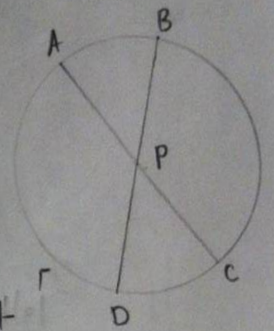
Physics Cup 2017

Problem 4

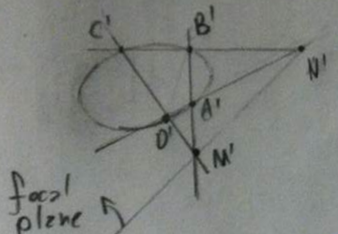


Our construction will be based on a couple of facts:
(1) The image of a straight line is also a straight line
(2) Parallel incident rays converge on a plane normal to the lens' axis: the focal plane.

Let Γ be the circle and Γ' its elliptical image, with P and P' being the circle's center and the respective image.



Consider a rectangle ABCD inscribed in Γ, and its image A'B'C'D'. From fact (1), $P \in \overline{AC}, \overline{BD} \Leftrightarrow P' \in \overline{A'C'}, \overline{B'D'}$.
If we now define $M' = \overleftrightarrow{A'B'} \cap \overleftrightarrow{C'D'}$ and $N' = \overleftrightarrow{A'D'} \cap \overleftrightarrow{C'B'}$, from fact (1) again, M' is the image of a point $M = \overleftrightarrow{AB} \cap \overleftrightarrow{CD}$.
Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, M is at infinity, and consequently all rays coming from it are parallel to $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ $\Rightarrow$ M' is in the focal plane.



By the same reasoning, N' is also in the focal plane, and therefore the focal plane is given by $\overleftrightarrow{M'N'}$.
Finally consider a pair of rays, one from M and one from N, that both go through the lens' center O. These rays are initially perpendicular (they come from intersections of the rectangle) and remain perpendicular after refraction (they go through the center) $\Rightarrow M'\hat{O}N' = 90°$.
$\Rightarrow$ The circumference of diameter $\overline{M'N'}$ goes through O. If we now choose a new rectangle EFGH, we can generate new points $R', Q' \in \overleftrightarrow{M'N'}$ through the same steps such that O is in the circumference of diameter $\overline{R'Q'}$. These circumferences intersect at two points. Since the image is real, we choose O to be the intersection on the side of $\overline{M'N'}$ without P'.