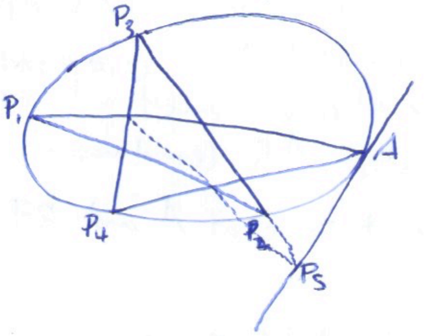
Physics Cup Q4 we first establish 3 construction lemmas:

① Given point A on ellipse E, we can construct the tangent to E at A.

Proof: choose 4 other points P_1, P_2, P_3, P_4.



consider $P_5 = (AP_1 \cap P_3P_4)(AP_4 \cap P_1P_2) \cap P_2P_3$.

Then by Pascal's theorem on conic E (an ellipse) using degenerate hexagon $AAP_1P_2P_3P_4$, $\underline{\underline{AP_5}}$ is tangent to E at A.

↳ we have just constructed tangent to E at A ✗

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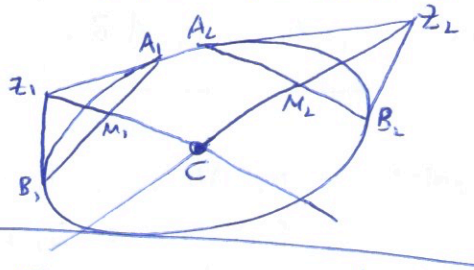
② Given ellipse E, we may construct its centre, C.

Proof: take 2 points A, B on ellipse E. Let their tangents (constructed by ①) meet at Z. If midpoint of AB is M, then ZM passes through the centre of E.

[This is because:

it is true when E is a circle by symmetry, and E can be transformed to a circle by an affine transformation (preserving collinearity and the midpoint and tangent conditions).]

Thus drawing ZM line for 2 choices of points (A, B) on E will cause them to intersect at the desired centre of E, $\underline{\underline{C}}$ ✗



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③ Given ellipse E at point O in the ellipse, we may construct line l passing through O, intersecting E at A and B such that O is the midpoint of AB.

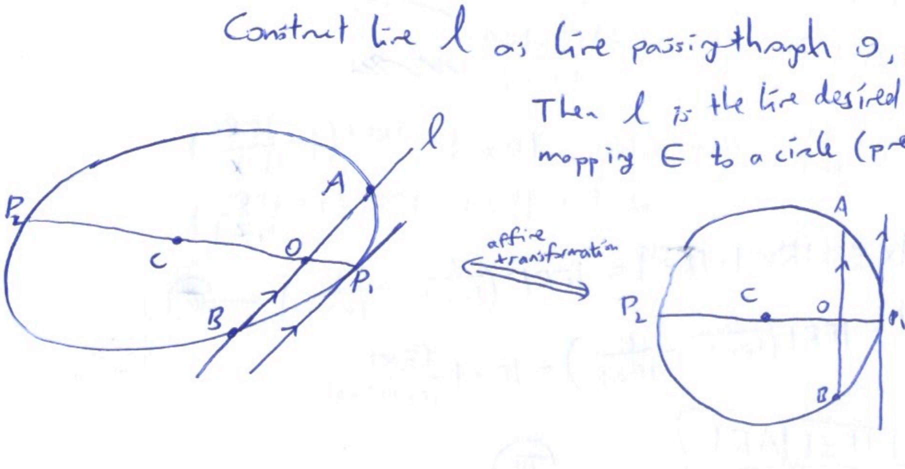
Proof: Construct centre C of ellipse E (using ②).

Let CO intersect E at P_1 and P_2; construct tangent to E at P_1 (or P_2) (using ①).

[Note: their tangents are parallel by symmetry].

Construct line l as line passing through O, parallel to tangent to E at P_1.

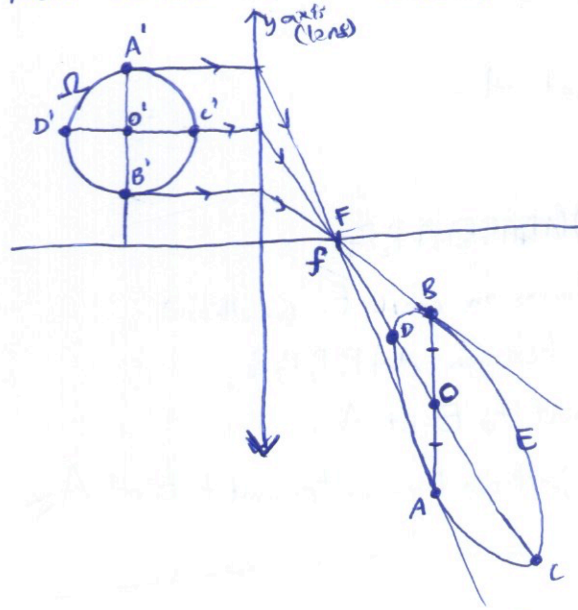
Then l is the line desired. This is because taking the affine transformation mapping E to a circle (preserving the parallel and midpoint and tangent conditions),



O is the midpoint of AB by symmetry.

(and line l through O is the unique choice). ✗

Now we consider the ~~image of~~ circle Ω about, lens with focus f that generated the image of E and the center image O.



Then it is easy to see that if A', C', B', D' and O' are the north, east, south, west and center of Ω,

and F is point of the focus opposite Ω,

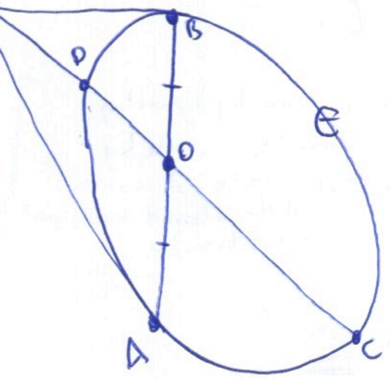
then by ray tracing, since A', O', B' are same distance from the lense and O' is midpoint of $A'B'$, $(A_0 B \,//\, y\text{axis})$

thus, their images A, O, B are all same distance from the lens and O is midpoint of AB (equal magnification).

Also AF is tangent to E at A and BF is tangent to E at B.

and D', O', C' collinear and parallel to X-axis (images correspondingly) thus by ray-tracing, F, D, O, C collinear.

We want to reconstruct points A, B, C, D and F from just the ~~e~~llipse E (image) and image of center, O.



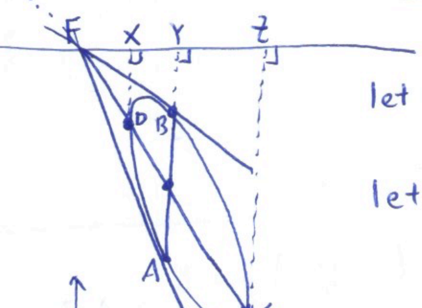
using ③, we construct points A, B such that O is midpoint of AB.

since this choice of AB is unique, thus A, B are the points as desired.

using ①, construct tangents to E at A and B and have them intersect at F.

~~using~~ now let line FO intersect E at D and C (D is closer to F than C).

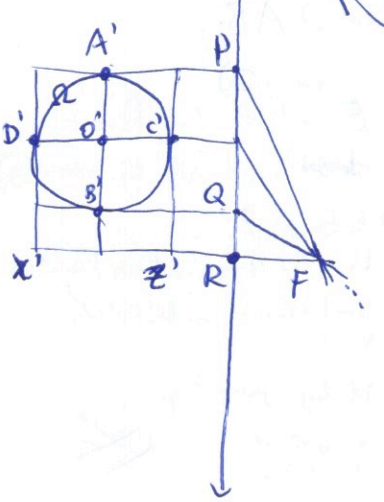
We also know $AB \,//\,$ lens. (y-axis). We can thus construct the x axis as the line through F perpendicular to AB.



let the points of projection of D, A/B and C to x-axis be X, Y and Z respectively.

let points of projection of D', C' to x-axis be X' and Z' respectively.

Then X', Z' form images X and Z through the lens.



let points of projection of B', A' to y-axis (lens) be Q, P respectively.

let origin be R.

Ω is circle $\therefore |PQ| = |X'Z'|$. — I

$\Delta PQF \sim \Delta ABF \therefore |PQ| = \dfrac{|AB| \cdot |FR|}{|FY|}$ ~~II~~

$\dfrac{1}{|RX'|} + \dfrac{1}{|RX|} = \dfrac{1}{|RF|}$ (lens equation) $\therefore |RX'| = |RF| \left(1 + \dfrac{|FR|}{|FX|}\right)$

similarly $|RZ'| = |RF| \left(1 + \dfrac{|FR|}{|FZ|}\right)$

$\therefore |X'Z'| = |RX'| - |RZ'| = |FR|^2 \left(\dfrac{1}{|FX|} - \dfrac{1}{|FZ|}\right)$ — II

I = II $\therefore \dfrac{|AB|}{|FY|} = |FR| \left(\dfrac{1}{|FX|} - \dfrac{1}{|FZ|}\right) = |FR| \dfrac{|ZX|}{|FX||FZ|}$

$$\therefore \boxed{|FR| = \frac{|FX|\,|FZ|\,|AB|}{|FY|\,|XZ|}}$$ — III

use this to construct point R

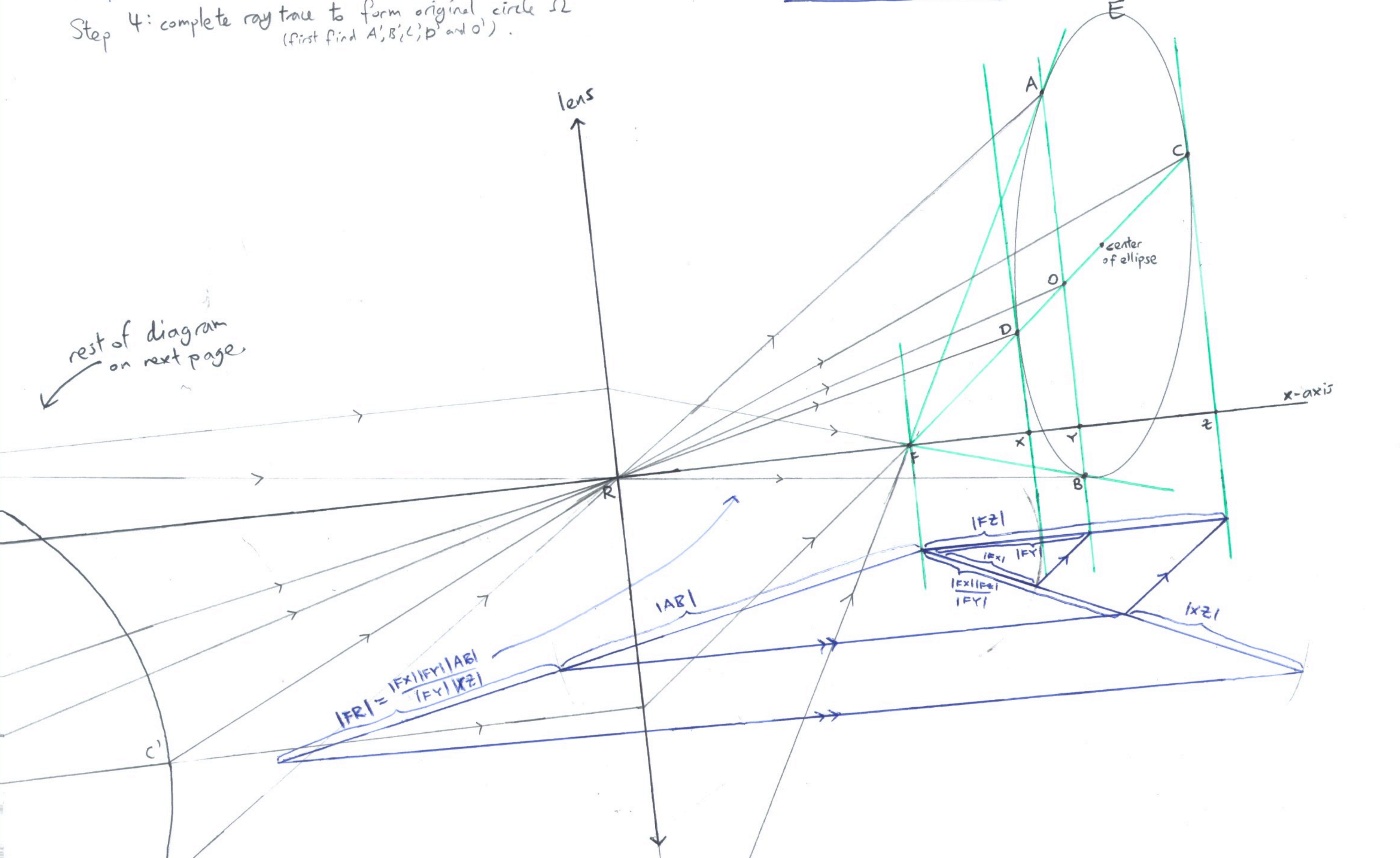
lens is constructed!

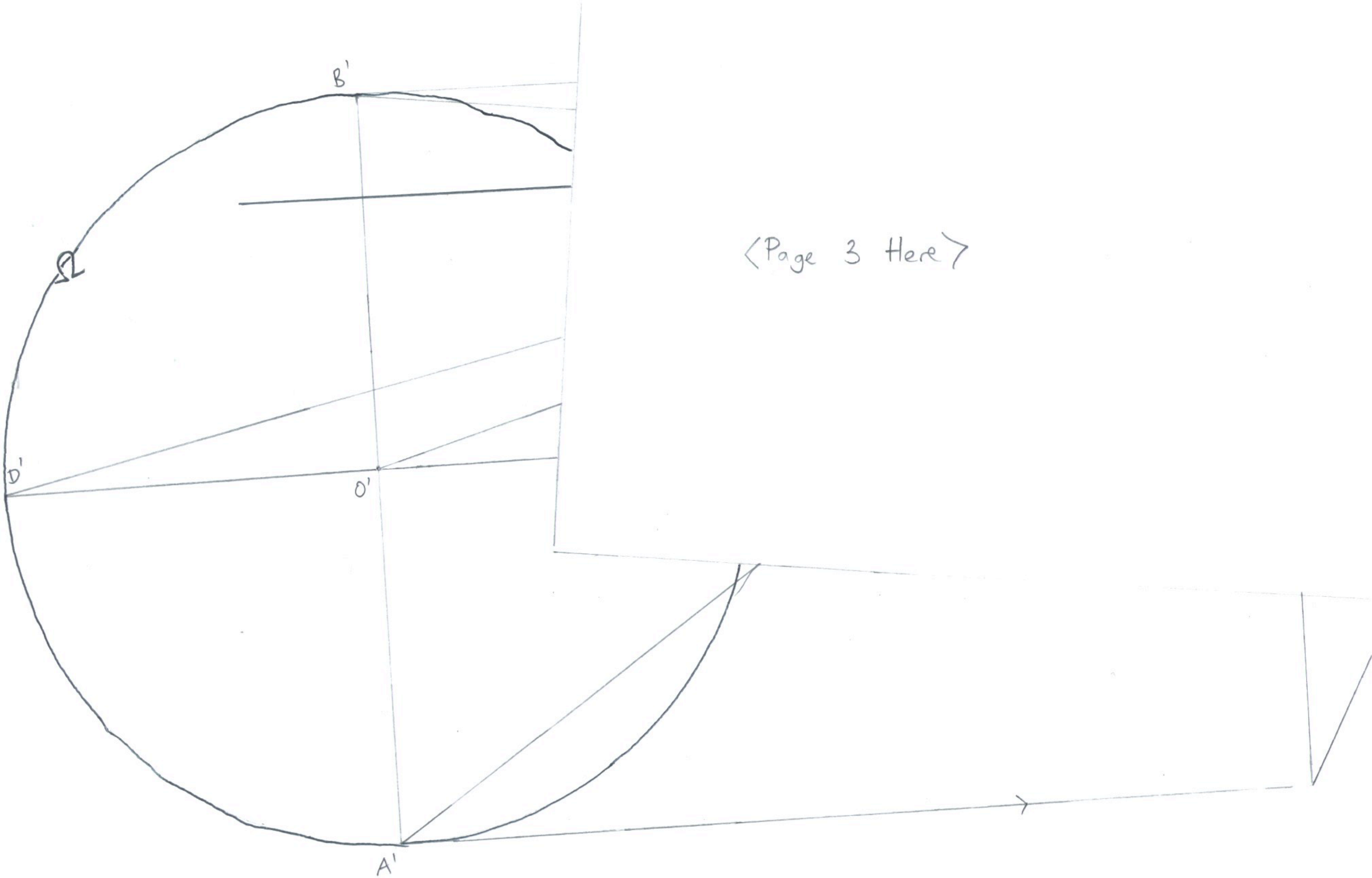
Step 1: construct A, B, C, D and F from ellipse E and centerimage O. (use ①, ② and ③).
(lens orientation determined)

Step 2: draw x-axis and projection points X, Y, Z.

Step 3: use equation III and the lengths on diagram to construct |FR|, and transfer it onto x-axis to determine position of lens.

Step 4: complete ray trace to form original circle Ω
(first find A', B', C', D' and O').

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B'
Ω
D'
O'
A'