Problem 01 - Diogo Correia Netto

- Motivation: find a charge distribution that produces a quadratic potential (in such a manner as to cancel the parallel external field of the punctual charge: $\vec{E}_{\parallel}^{\text{charge}} \propto \vec{r}$ in the limit of $r \ll L$).

- According to the hint given for problem 1 (which is included in the end of this document), a uniformly charged ellipsoid produces an electric potential that is quadratic in the coordinates.

- Equation of an ellipsoid with semi axis a, a and c:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1. \tag{1}$$

- Formula for the potential of a charged body:

$$V(x, y, z) = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho(x', y', z')dx'dy'dz'}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}. \tag{2}$$

- Let's consider a uniformly charged ellipsoid with semi axis a, a and $c \to 0$, obtaining in this limiting case a simple disc of radius a (which will henceforth be renamed as the radius of the disc, because r will be used as the radial cylindrical coordinate).

  In this limit $(z - z')^2 \ll (x - x')^2, (y - y')^2$, then

$$\Rightarrow V(x, y) = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho dx'dy'dz'}{\sqrt{(x - x')^2 + (y - y')^2}} \tag{3}$$

  According to equation (1) the limits of integration for z' are $\pm c\sqrt{1 - \left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{a}\right)^2}$.

  As the density ρ is assumed to be uniform:

$$\Rightarrow V(x, y) \approx \frac{\rho c}{2\pi\varepsilon_0} \iint \frac{dx'dy'\sqrt{1 - \frac{x'^2 + y'^2}{a^2}}}{\sqrt{(x - x')^2 + (y - y')^2}}. \tag{4}$$

- Let's make the substitution $\left| \begin{array}{l} x' = k\cos\phi \\ y' = k\sin\phi \end{array} \right\} k \leq a.$

  The element of area is now written as:

$$dA = dx'dy' = kdkd\phi. \tag{5}$$

$$\sqrt{1-\frac{x'^2+y'^2}{a^2}} = \sqrt{1-\frac{k^2}{a^2}}, \tag{6}$$

$$\Rightarrow V(x,y) \approx \frac{\rho c}{2\pi\varepsilon_0}\int_0^a\int_0^{2\pi}\frac{kdk\sqrt{1-\frac{k^2}{a^2}}}{\sqrt{x^2+y^2+k^2-2kx\cos\phi-2ky\sin\phi}}. \tag{7}$$

- As we already know, the expansion of the potential is quadratic in the coordinates

$$\Rightarrow V(x,y) = A + B(x^2+y^2), \tag{8}$$

where A and B are constants.

- Let's calculate the values of A and B:

$\rightarrow A$: suppose $x = y = 0$.

$$V(0,0) = \frac{\rho c}{2\pi\varepsilon_0}\int_0^a\int_0^{2\pi} dkd\phi\sqrt{1-\frac{k^2}{a^2}} = A + B(0^2+0^2)$$

$$\Rightarrow A = \frac{\pi^2\rho c a}{4\pi\varepsilon_0}. \tag{9}$$

$\rightarrow B$: to calculate B, let's make $\left|\begin{array}{l} x = a \\ y = 0 \end{array}\right.$,

$$\Rightarrow V(a,0) = \frac{\rho c}{2\pi\varepsilon_0}\int_0^a\int_0^{2\pi}\frac{kdkd\phi\sqrt{1-\frac{k^2}{a^2}}}{\sqrt{k^2-2ka\cos\phi+a^2}}. \tag{10}$$

- As $k \leq a$, the denominator can be expanded in terms of Legendre Polynomials (P_j):

$$\rightarrow \frac{1}{\sqrt{k^2-2ka\cos\phi+a^2}} = \frac{1}{a}\sum_{j=0}^{\infty}\left(\frac{k}{a}\right)^j P_j(\cos\phi), \tag{11}$$

$$V(a,0)=\frac{\rho c}{2\pi\varepsilon_0 a}\int_0^a\int_0^{2\pi}kdkd\phi\sqrt{1-\frac{k^2}{a^2}}\sum_{j=0}^{\infty}\left(\frac{k}{a}\right)^j P_j(\cos\phi) \tag{12}$$

- Let's define

$$\alpha_j\equiv\int_0^{2\pi}P_j(\cos\phi)d\phi \tag{13}$$

$$\Rightarrow V(a,0)=\frac{\rho c}{2\pi\varepsilon_0 a}\int_0^a kdk\sqrt{1-\frac{k^2}{a^2}}\sum_{j=0}^{\infty}\left(\frac{k}{a}\right)^j\alpha_j. \tag{14}$$

- Defining $\tilde{y}\equiv\dfrac{k}{a}$,

$$\Rightarrow V(a,0)=\frac{\rho ca}{2\pi\varepsilon_0}\sum_{j=0}^{\infty}\int_0^1\tilde{y}d\tilde{y}\sqrt{1-\tilde{y}^2}\tilde{y}^j\alpha_j. \tag{15}$$

- Let

$$\eta\equiv\sum_{j=0}^{\infty}\int_0^1 d\tilde{y}\sqrt{1-\tilde{y}^2}\tilde{y}^{j+1}\alpha_j \tag{16}$$

a constant to be determined.

- $V(a,0)=\dfrac{\rho ca}{2\pi\varepsilon_0}\eta=A+B(a^2+0^2)$

$$\Rightarrow\frac{\rho ca\eta}{2\pi\varepsilon_0}=A+Ba^2\quad\Rightarrow\quad Ba^2=\frac{\rho ca\eta}{2\pi\varepsilon_0}-A,$$

using A given by eq. (9), we have

$$\Rightarrow Ba^2=\frac{\rho ca}{4\pi\varepsilon_0}\left(2\eta-\pi^2\right)$$

$$B=\frac{\rho c}{4\pi\varepsilon_0 a}\left(2\eta-\pi^2\right) \tag{17}$$

- Considering that $Q=\rho V$ where $V=\frac{4}{3}\pi a^2c$ is the volume of the ellipsoid

$$\Rightarrow Q=\rho\left(\frac{4}{3}\pi a^2 c\right) \tag{18}$$

$$\Rightarrow V(x,y) = A - \frac{\beta Q(x^2+y^2)}{\varepsilon_0 a^3}, \tag{19}$$

where

$$\beta = \frac{3(\pi^2 - 2\eta)}{16\pi^2}$$

is a constant.

- It should be noted that if $Q > 0$ the parallel electric field is directed along the $+\hat{r}$ direction.

  Hence, $\beta > 0$.

- Now let's calculate the electric field produced by the potential given in eq. (19)

$$\vec{E}_1 = -\vec{\nabla} V = \frac{2\beta Q \vec{r}}{\varepsilon_0 a^3},$$

where $\vec{r} = \sqrt{x^2+y^2}\,\hat{r}$.

- The condition $\vec{E}_\parallel = 0$ for a conductor says:

$$\vec{E}_\parallel = 0 \quad \Rightarrow \quad \vec{E}_1 + \vec{E}_\parallel^{\text{charge}} = 0.$$

But $\vec{E}_\parallel^{\text{charge}} = \dfrac{q\vec{r}}{4\pi\varepsilon_0 \left(L^2+r^2\right)^{3/2}} \approx \dfrac{q\vec{r}}{4\pi\varepsilon_0 L^3}$.

Thus:

$$\frac{2\beta Q\vec{r}}{\varepsilon_0 a^3} + \frac{q\vec{r}}{4\pi\varepsilon_0 L^3} = 0 \quad \Rightarrow \quad Q = -\frac{qa^3}{8\pi\beta L^3}. \tag{20}$$

- Let's make a comparison:

$$\begin{aligned} V(x,y) = &\ \frac{\rho c}{2\pi\varepsilon_0}\iint \frac{dx'dy'\sqrt{1-\left(\frac{x'}{a}\right)^2-\left(\frac{y'}{a}\right)^2}}{\sqrt{(x-x')^2+(y-y')^2}} \\ = &\ \frac{1}{4\pi\varepsilon_0}\iint \frac{\sigma_1 dA}{\sqrt{(x-x')^2+(y-y')^2}} \end{aligned}$$

We conclude that ellipsoidal distribution is equivalent to a superficial charge

$$\sigma_1(x', y') = 2\rho c\sqrt{1 - \left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{a}\right)^2} \tag{21}$$

Using that $V = \frac{4}{3}\pi a^2 c$, $Q = \rho V$ and $x'^2 + y'^2 = k^2$:

$$\sigma_1(k) = \frac{3Q}{2\pi a^2}\sqrt{1 - \frac{k^2}{a^2}} = -\frac{3qa^3}{16\pi^2 a^2 \beta L^3}\sqrt{1 - \frac{k^2}{a^2}}.$$

In terms of the radial coordinate r:

$$\sigma_1(r) = -\frac{3qa}{16\pi^2\beta L^3}\sqrt{1 - \frac{r^2}{a^2}}. \tag{22}$$

- To keep the disc neutral and equipotential, we must distribute a charge $-Q$ in such a manner as to not produce any parallel field. This is achieved by projecting a spherical shell onto the disc (this claim will be proved in the end of solution)

$$\Rightarrow \sigma_2(r) = -\frac{Q}{2\pi a\sqrt{a^2 - r^2}} = \frac{qa^2}{16\pi^2\beta L^3\sqrt{a^2 - r^2}}, \tag{23}$$

where σ_2 is due to the projection of the spherical shell.

- The total charge is given by:

$$\sigma_{\text{total}}(r) = \sigma_1(r) + \sigma_2(r) = -\frac{3qa}{16\pi^2\beta L^3}\sqrt{1 - \frac{r^2}{a^2}} + \frac{qa^2}{16\pi^2\beta L^3\sqrt{a^2 - r^2}}. \tag{24}$$

- Due to the azimuthal symmetry of the problem the electric force is perpendicular to the plane of the disc. Then, the electric force is given by:

$$F = \int \sigma(r) dA \cdot E_n, \tag{25}$$

where

$$E_n = -\frac{qL}{4\pi\varepsilon_0(L^2 + r^2)^{3/2}}. \tag{26}$$

$$F = \frac{qa}{16\pi^2\beta L^3}\cdot\frac{qL}{4\pi\varepsilon_0}\int_0^a\left[3\sqrt{1-\frac{r^2}{a^2}}-\frac{1}{\sqrt{1-\frac{r^2}{a^2}}}\right]\cdot\frac{2\pi r dr}{(L^2+r^2)^{3/2}}$$

$$= \frac{q^2aL}{32\pi^2\varepsilon_0\beta L^3}\int_0^a\left[3\sqrt{1-\frac{r^2}{a^2}}-\frac{1}{\sqrt{1-\frac{r^2}{a^2}}}\right]\cdot\frac{r dr}{(L^2+r^2)^{3/2}}$$

$$= \frac{q^2aL}{32\pi^2\varepsilon_0\beta L^3}\int_0^a\left[3\sqrt{1-\frac{r^2}{a^2}}-\frac{1}{\sqrt{1-\frac{r^2}{a^2}}}\right]\cdot\frac{\left(\frac{r}{a}\right)d\left(\frac{r}{a}\right)\left(a\cdot\frac{1}{a}\right)}{a\left(\frac{L^2}{a^2}+\frac{r^2}{a^2}\right)^{3/2}}$$

In terms of $\gamma = r/a$ and $\delta = L/a$

$$F = \frac{q^2L}{32\pi^2\varepsilon_0\beta L^3}\int_0^1\left[3\sqrt{1-\gamma^2}-\frac{1}{\sqrt{1-\gamma^2}}\right]\cdot\frac{\gamma d\gamma}{(\gamma^2+\delta^2)^{3/2}}. \tag{27}$$

According to Wolfram Alpha:

$$I = \int_0^1\left[3\sqrt{1-\gamma^2}-\frac{1}{\sqrt{1-\gamma^2}}\right]\cdot\frac{\gamma d\gamma}{(\gamma^2+\delta^2)^{3/2}}$$

$$= \left[3\tan^{-1}\left(\frac{1-\gamma^2}{\gamma^2+\delta^2}\right)-\left(\frac{3\delta^2+2}{\delta^2+1}\right)\left(\frac{1-\gamma^2}{\delta^2+\gamma^2}\right)^{1/2}\right]_0^1$$

$$\Rightarrow I = \frac{3\delta^2+2}{\delta^3+\delta}-3\tan^{-1}\left(\frac{1}{\delta}\right) \tag{28}$$

- Expanding I for $\delta \to +\infty$ (As $L \gg a$)

$$I \approx \frac{2}{5\delta^5}-\frac{4}{7\delta^7}. \tag{29}$$

$$\Rightarrow F \approx \frac{q^2}{32\pi^2\varepsilon_0\beta L^2}\left(\frac{2}{5\delta^5}-\frac{4}{7\delta^7}\right)$$

Neglecting terms of $\mathcal{O}(1/\delta^7)$:

$$F \approx \frac{q^2 a^5}{80\pi^2 \varepsilon_0 \beta L^7}. \tag{30}$$

- To end our solution we must calculate the value of β. Let's make some iterations to find η:

N	α_n	$J_n = \int_0^1 dy\sqrt{1-y^2}y^{n+1}$	$\alpha_n J_n$
0	2π	$1/3$	$2\pi/3$
2	$\pi/2$	$2/15$	$\pi/15$
4	$9\pi/32$	$8/105$	$3\pi/140$
6	$100\pi/256$	$16/315$	$5\pi/252$
8	$1225\pi/8192$	$128/3465$	$35\pi/6336$
10	$3969\pi/32768$	$256/9009$	$63\pi/18304$

and $\alpha_n = 0$ for all n odd.

Summing the terms $\alpha_n J_n$:

$$\eta \approx 2.462$$

$$\Rightarrow \beta = \frac{3(\pi^2 - 2\eta)}{16\pi^2} \approx 0.09397$$

After 150 iterations in *Mathematica*, we obtain $\eta = 2.4674$ and $\beta = 0.09375$. As we increase the number of steps, beta gets closer to 0.09375. Therefore, we conclude that $\beta \approx 0.09375 = 3/32$.

The expression for the force is

$$F \approx \frac{q^2 a^5}{80\pi^2 \varepsilon_0 \beta L^7}, \tag{31}$$

where $\beta \approx 0.09375 = 3/32$.

Proof of the claim about the projection of the spherical shell

We already know that the field inside a spherical shell is zero. Let's consider two charges q_1 and q_2 as shown in figure 1.

As the field in the shell is zero, the pairs of opposite charges give equal (but opposite) fields. Now, the projections give equal and opposite fields if:

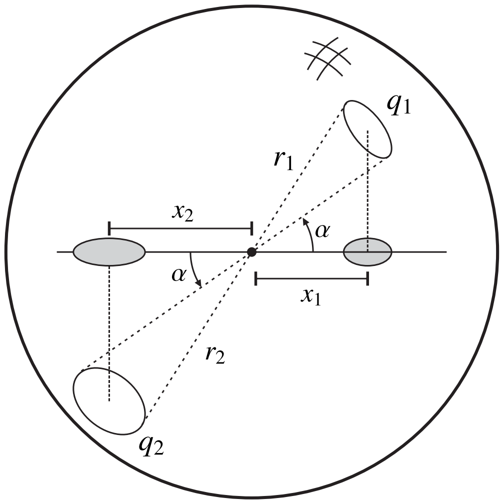


Figure 1: Projection of charges in the spherical shell.

$$\frac{q_1}{4\pi\varepsilon_0 x_1^2} = \frac{q_2}{4\pi\varepsilon_0 x_2^2}, \tag{32}$$

which is true because $\left| \begin{array}{l} x_1 = r_1 \cos\alpha \\ x_2 = r_2 \cos\alpha \end{array} \right.$.

Proof of the volume of an ellipsoid

Let's start with the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

We know that an ellipsoid of circular section of radius a and semi axis b is obtained by rotating the curve $y = f(x)$ around the y axis.

Thus

$$V = \int_{y_0}^{y_1} \pi x^2 dy, \tag{33}$$

As $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow x^2 = \dfrac{a^2}{b^2}\left(b^2 - y^2\right)$, we have

$$\begin{aligned} V = &\ \pi \int_{-b}^{b} \frac{a^2}{b^2}\left(b^2 - y^2\right) dy = \frac{\pi a^2}{b^2} \int_{-b}^{b} \left(b^2 - y^2\right) dy = \frac{2\pi a^2}{b^2} \int_{0}^{b} \left(b^2 - y^2\right) dy \\ = &\ \frac{2\pi a^2}{b^2} \left[b^2 \cdot y - \frac{y^3}{3}\right]_0^b = \frac{2\pi a^2}{b^2}\left[b^3 - \frac{b^3}{3}\right] = \frac{2\pi a^2}{b^2}\left(\frac{2b^3}{3}\right) \end{aligned}$$

$$\Rightarrow V = \frac{4}{3}\pi a^2 b \tag{34}$$

Physics Cup 2017 - Problem 1 with hints. 15th April 2017

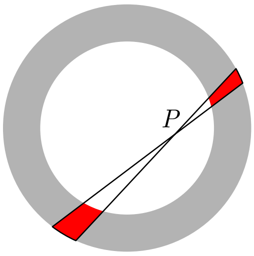
Estimate by the order of magnitude the interaction force between a point charge q and a circular metallic disc of radius r if the charge is at the axis of the disc, and the distance between the disc and the charge is $L \gg r$. The total charge of the disc is 0 and the thickness of the disc is negligibly small.

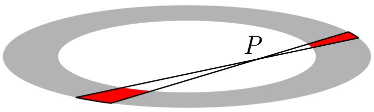
Hints: *First,* notice that the standard electrical image method fails here because the image charge would be in the same region of space where we look for the solution (if you are confused about this statement, read more at
http://www.ipho2012.ee/physicscup/physics-solvers-mosaic/5-images-or-roulette/

Second, notice that the total charge on the disc is zero, but it must re-distribute (creating positive and negative charge areas) so as to compensate for the electric field of the point charge and ensure that the disc remains equipotential.

Third, if you want to find the exact answer, you'll find it useful to know that inside an ellipsoid with homogeneous constant volume charge density, the electric potential is a quadratic polynomial of the coordinates.

The proof of this fact has several steps. It starts with an observation that inside an ellipsoidal shell of constant volume charge density, electric field is zero. [Ellipsoidal shell is what you obtain if a spherical shell is by compressed, i.e. an affine transformation is made, along certain direction(s).] If you take an arbitrary point P inside an ellipsoidal shell, it is fairly easy to see that the contributions of opposing pieces of the shell (which you obtain if you cut the shell with a cone of very small tip angle α) cancel out, hence the field is zero.





Now, let us consider two similar ellipsoids, one large, of length A, denoted as E_l, and one very small, of length a, denoted as E_s, both centred at the origin.

The second step is using the similarity consideration: we can say that if the small charged ellipsoid (of constant volume charge density ρ) has potential distribution $\varphi(\vec{r})$ for $\vec{r} \in E_s$ then the large charged ellipsoid must have potential distribution $\left(\frac{A}{a}\right)^2 \varphi(\vec{r}\frac{a}{A})$ for $\vec{r} \in E_l$. On the other hand, if we consider for a large charged ellipsoid a point $\vec{r}_0$ so close to the origin that it falls also into the small ellipsoid (i.e. $\vec{r}_0 \in E_s$), these charges of the large ellipsoid which remain outside E_s (and form a thick ellipsoidal shell) give zero field and no contribution to the potential inside E_s (assuming that the origin defines the zero potential level). So, the potential at such $\vec{r}_0$ is contributed only by the charges inside E_s, i.e. $\left(\frac{A}{a}\right)^2 \varphi(\vec{r}_0\frac{a}{A}) = \varphi(\vec{r}_0)$.

Finally, as the last step of the proof, notice that the potential is clearly a smooth and continuous function of coordinates and can be expanded into Taylor series near its centre; at very small distances, the main terms of the series dominate over the higher order terms so that for $\vec{r}_0$ very close to the origin — much closer than the size of the ellipsoid —, $\varphi(\vec{r}_0)$ is a quadratic polynomial of the coordinates. This means that due to the property $\left(\frac{A}{a}\right)^2 \varphi(\vec{r}_0\frac{a}{A}) = \varphi(\vec{r}_0)$, it is also a quadratic polynomial everywhere inside the ellipsoid (because we can use arbitrarily small $|\vec{r}_0|$ with arbitrarily large $\frac{A}{a}$).

Results thus far (by the order of submission):
Kaarel Hänni: 2.5937
Marco Malandrone: 2.3579
Siddharth Tiwary: 1.9292

Non-official participants:
Taavet Kalda: 1.9292