Physics Cup 2017 - Problem 1 with hints. 7th May 2017

Estimate by the order of magnitude the interaction force between a point charge q and a circular metallic disc of radius r if the charge is at the axis of the disc, and the distance between the disc and the charge is $L \gg r$. The total charge of the disc is 0 and the thickness of the disc is negligibly small.

Hints: *First,* notice that the standard electrical image method fails here because the image charge would be in the same region of space where we look for the solution (if you are confused about this statement, read more at http://www.ipho2012.ee/physicscup/physics-solvers-mosaic/5-images-or-roulette/

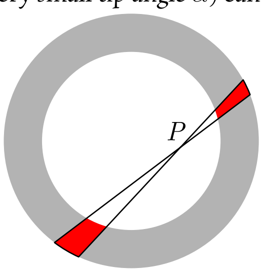
Second, notice that the total charge on the disc is zero, but it must re-distribute (creating positive and negative charge areas) so as to compensate for the electric field of the point charge and ensure that the disc remains equipotential.

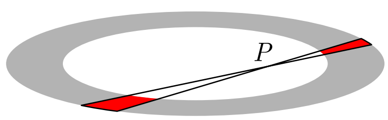
Third, notice that the disc must be equipotential, hence the potential difference created by the induced charges must be of the same order of magnitude as the one created by the remote charge q (this condition makes it possible to estimate the magnitude of the induced charges $\pm Q$). Alternatively one can consider the tangential component of the electric field near the disc surface: the component due to the remote charge q must be balanced by the field created by the ring charge $+Q$ and the charge $-Q$ spread over the central region of the disc. This condition makes it possible to estimate Q, and if we know Q, we can estimate the force.

Fourth, if you want to find the exact answer, you'll find it useful to know that inside an ellipsoid with homogeneous constant volume charge density, the electric potential is a quadratic polynomial of the coordinates.

Fifth, of course you should go to the limit of very thin ellipsoids — the smallest semiaxis is much smaller than the two larger and equal ones.

The proof of the fact provided by 4th hint has several steps. It starts with an observation that inside an ellipsoidal shell of constant volume charge density, electric field is zero. [Ellipsoidal shell is what you obtain if a spherical shell is by compressed, i.e. an affine transformation is made, along certain direction(s).] If you take an arbitrary point P inside an ellipsoidal shell, it is fairly easy to see that the contributions of opposing pieces of the shell (which you obtain if you cut the shell with a cone of very small tip angle α) cancel out, hence the field is zero.





Now, let us consider two similar ellipsoids, one large, of length A, denoted as E_l, and one very small, of length a, denoted as E_s, both centred at the origin.

The second step is using the similarity consideration: we can say that if the small charged ellipsoid (of constant volume charge density ρ) has potential distribution $\varphi(\vec{r})$ for $\vec{r} \in E_s$ then the large charged ellipsoid must have potential distribution $\left(\frac{A}{a}\right)^2 \varphi(\vec{r}\frac{a}{A})$ for $\vec{r} \in E_l$. On the other hand, if we consider for a large charged ellipsoid a point $\vec{r}_0$ so close to the origin that it falls also into the small ellipsoid (i.e. $\vec{r}_0 \in E_s$), these charges of the large ellipsoid which remain outside E_s (and form a thick ellipsoidal shell) give zero field and no contribution to the potential inside E_s (assuming that the origin defines the zero potential level). So, the potential at such $\vec{r}_0$ is contributed only by the charges inside E_s, i.e. $\left(\frac{A}{a}\right)^2 \varphi(\vec{r}_0\frac{a}{A}) = \varphi(\vec{r}_0)$.

Finally, as the last step of the proof, notice that the potential is clearly a smooth and continuous function of coordinates and can be expanded into Taylor series near its centre; at very small distances, the main terms of the series dominate over the higher order terms so that for $\vec{r}_0$ very close to the origin — much closer than the size of the ellipsoid —, $\varphi(\vec{r}_0)$ is a quadratic polynomial of the coordinates. This means that due to the property $\left(\frac{A}{a}\right)^2 \varphi(\vec{r}_0\frac{a}{A}) = \varphi(\vec{r}_0)$, it is also a quadratic polynomial everywhere inside the ellipsoid (because we can use arbitrarily small $|\vec{r}_0|$ with arbitrarily large $\frac{A}{a}$).

Results thus far (by the order of submission):
Kaarel Hänni: 2.5937
Marco Malandrone: 2.3579
Siddharth Tiwary: 1.9292
Diogo Netto: 1.7538
Faizal Husni: 1.7715
Victor Almeida Ivo: 1.6105
Akihiro Watanabe: 1.4641
Gabriel Golfetti: 1.331
Non-official participants (by the order of submission):
Taavet Kalda: 1.9292
Elvinas Ribinskas: 1.9487