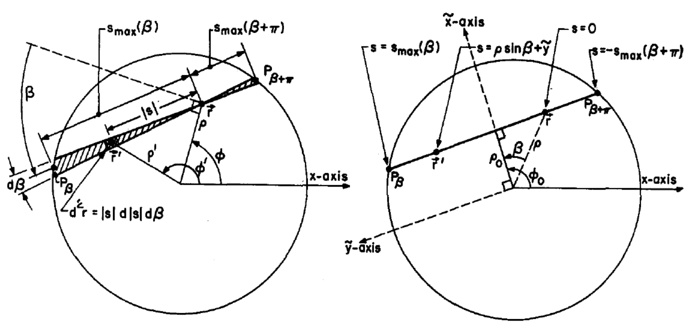
The potential at point $\vec{r}$ on the disc can be written as a function of surface charge distribution

$$V_{(r)} = \int \sigma_{(r')} \frac{d^2 r'}{|\vec{r} - \vec{r'}|}$$



From the figure we can write the integral as (where $|s| = |\vec{r} - \vec{r'}|$)

$$V_{(r)} = \int_0^{2\pi} \int_0^{s_{max}} \sigma d|s| d\beta \tag{1}$$

We can use s instead of $|s|$ if we change the range of θ to $\{0, \pi\}$. For constant θ, integral represents the contribution of a single wedge shaped area. But the specific shape of this area doesn't change the integral because the integral is only a function of the length of the area and the surface charge density. So we can take this area as a chord that is charged

$$\lambda = \int \sigma_{r'} ds \tag{2}$$

This chord can be determined by its midpoints coordinates $\{\rho_0, \phi_0\}$. From the figure we see that

$$\rho_0 = \rho cos\beta \tag{3}$$

And because of the cylindrical symmetry of the problem λ is not a function of ϕ_0. We can take $\phi_0 = 0$ from now on. Then our coordinates takes the form $x = \rho \cos\beta, y = \rho \sin\beta$ and equation (2) will be identical with

$$\lambda = \int \sigma dy \tag{4}$$

Since they both are the charge on the chord. So we can write the potential as

$$V_{(r)} = \int_0^{\pi} \lambda_{(\rho cos\beta)} d\beta \tag{5}$$

From this equation we see that if we want $V_{(r)}$ to be constant then

$$\lambda_{(\rho cos\beta)} = \lambda_0$$
$$V_{(r)} = \pi\lambda_0$$

But before using this charge distribution for λ we need to show that this line charge is the result of a surface charge density which only depends on coordinate ρ. So to show that we use a spherical shell which has constant surface charge across its surface. If we smash the shell flat on the x-y plane then it is obvious that the surface charge will be symmetric. To find λ we need to find net charge on the sphere which is between x and $x+dx$ because from definition $dq = \lambda_{(x)}dx$ Net charge at this segment can be calculated as

$$dq = (\frac{Q}{4\pi a^2})(2\pi a cos\theta)ad\theta \tag{6}$$

Where $d(asin\theta) = dx$. From $\frac{dq}{dx} = \lambda$ we get

$$\lambda = \frac{Q}{2a} = const. \tag{7}$$

So we see that it is indeed possible to have $\lambda = \lambda_0$ where surface charge is symmetrical.

Also the potential above the disc on the z-axis can be written using strip function as(where $x = \rho\cos\beta$)

$$V_{(z)} = \int_{-a}^{a} \lambda_{(x)} \frac{zdx}{z^2 + x^2} \tag{8}$$

This can be proved by putting equation (4) into (8)

$$\begin{aligned} V_{(z)} &= \int_{disk} dA \frac{z\sigma_{(x^2+y^2)}}{x^2 + z^2} \\ &= \int_0^a \rho d\rho \quad 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\phi z\sigma_{(\rho)}}{z^2 + \rho^2\cos^2\phi} \end{aligned}$$

Now we substitude

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\phi}{1 + (\frac{\rho}{z})^2 cos^2\phi} = \frac{\pi}{\sqrt{1 + (\frac{\rho}{z})^2}}$$

Which gives

$$V_{(z)} = \int_0^a 2\pi\rho d\rho \frac{\sigma_{(\rho)}}{\sqrt{x^2 + z^2}}$$

These steps can be reversed to obtain equation (8). We now have all the preliminary to deal with the charge q.

The charge q produces a potential

$$V_{ext}(\rho) = \frac{q}{\sqrt{L^2 + \rho^2}} \tag{9}$$

over the disc. Which must be compansated by the induced charges by the equation

$$V_{(\rho)} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \lambda_{(\rho \cos\beta)} d\beta$$

We can find λ by the use of the identity

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\beta}{1 + u^2 cos^2\beta} = \frac{\pi}{\sqrt{1+u^2}} \tag{10}$$

where $u = \frac{\rho}{L}$

If we substitute (10) into (9) we get

$$\lambda_{(x)} = -\frac{q}{\pi}\frac{L}{x^2 + L^2} \quad ; \quad x = \rho\cos\beta \tag{11}$$

Having found the λ we can find the total induced charge Q as

$$Q = \int_{-a}^{a} \lambda_{(x)} dx = -\frac{qL}{\pi}\int_{-a}^{a}\frac{dx}{x^2 + L^2} = -\frac{2q}{\pi}\arctan\frac{a}{L} \tag{12}$$

We can find potential of this charge distribution on the z axis according to the formula (8)

$$\begin{aligned} V(z) &= V(z, L) \\ &= -\frac{qzL}{\pi}\int_{-a}^{a}\frac{dx}{(L^2 + x^2)(z^2 + x^2)} \\ &= -\frac{qzL}{\pi(z^2 - L^2)}\int_{-a}^{a}(\frac{1}{L^2 + x^2} - \frac{1}{z^2 + x^2})dx \\ &= -\frac{2q}{\pi(z^2 - L^2)}[z\arctan\frac{a}{L} - L\arctan\frac{a}{z}] \end{aligned}$$

For the disc to be neutral a total charge of $-Q$ must redistribute over the disc according to the formula (7) so that the net potential on the z axis will be

$$V(z, L) = \frac{2q}{\pi(z^2 - L^2)}(L\arctan\frac{a}{z} - z\arctan\frac{a}{L}) + \frac{2q}{a\pi}\arctan\frac{a}{L}\arctan\frac{a}{z} \tag{13}$$

$$F = -q\frac{\partial V_{(z,L)}}{\partial z}\Big|_{z=L}$$

To find the force let us calculate the derivative of first term in the (13) as

$$V_1(z, L) = \frac{2q}{\pi(z^2 - L^2)}(L\arctan\frac{a}{z} - z\arctan\frac{a}{L}) \tag{14}$$

$$V_1(z, L) = V_1(L, z) \tag{15}$$

$$\tag{16}$$

$$dV_1(z, L) = \frac{\partial V_1(z, L)}{\partial z}dz + \frac{\partial V_1(z, L)}{\partial L}dL \tag{17}$$

z and L can be considered as mathemathical variables. Due to equation (15)

$$dV_1(z,L) = \frac{\partial V_1(z,L)}{\partial z}(dz+dL)$$
$$dV_1(L,L) = \frac{\partial V_1(z,L)}{\partial z}\Big|_{z=L} 2dL$$

Thus we can calculate the derivate of the first term as

$$F_1 = -q\frac{\partial V_1(z,L)}{\partial z}\Big|_{z=L}$$
$$= -\frac{q}{2}\frac{dV_1(L,L)}{dL}$$

In the limit $\lim_{z\to L}$ equation (14) takes form using L'Hôpital's rule

$$V_1(L,L) = \lim_{z\to L} \quad \frac{2q}{\pi(z^2-L^2)}(L\arctan\frac{a}{z} - z\arctan\frac{a}{L})$$
$$V_1(L,L) = -\frac{2q}{\pi}\frac{\frac{aL}{a^2+L^2}+\arctan\frac{a}{L}}{2L} = -\frac{q}{\pi}\left(\frac{a}{a^2+L^2}+\frac{\arctan\frac{a}{L}}{L}\right)$$

Now F_1 finally gives

$$F_1 = \frac{q^2}{2\pi}\left(\frac{a}{a^2+L^2}+\frac{\arctan\frac{a}{L}}{L}\right)'$$
$$= -\frac{q^2}{2\pi L^2}\left(\arctan\frac{a}{L}+\frac{a(3+\frac{a^2}{L^2})}{L(1+\frac{a^2}{L^2})^2}\right)$$

The derivative of the second term in equation (13) is considerably more trivial to find.

$$F_2 = -q\frac{\partial\frac{2q}{a\pi}\arctan\frac{a}{L}\arctan\frac{a}{z}}{\partial z}\Big|_{z=L}$$
$$= \frac{2q^2}{\pi}\frac{\arctan\frac{a}{L}}{a^2+L^2}$$

The net force is thus

$$F = F_1 + F_2 = \frac{q^2}{2\pi L^2}\left(\frac{\arctan u(3-u^2)}{1+u^2} - \frac{u(3+u^2)}{(1+u^2)^2}\right) \tag{18}$$

Where $u = \frac{a}{L}$. To use the condition $u \ll 1$ we shall expand the function

$$f_{(u)} = \frac{\arctan u(3-u^2)}{1+u^2} - \frac{u(3+u^2)}{(1+u^2)^2}$$

at $u = 0$ in tailor series which gives us

$$f_{(u)} = -\frac{16}{15}u^5 + \frac{256}{105}u^7 - \frac{416}{105}u^9 + \frac{19328}{3465}u^{11} + ...$$

Taking only the first term gives us the total force as

$$F = -\frac{8q^2a^5}{15\pi L^7} \tag{19}$$

In SI units

$$F = -\frac{2q^2a^5}{15\pi^2\epsilon_0 L^7} \tag{20}$$

References;

‡ American Journal of Physics 61, (1993);R. Friedberg