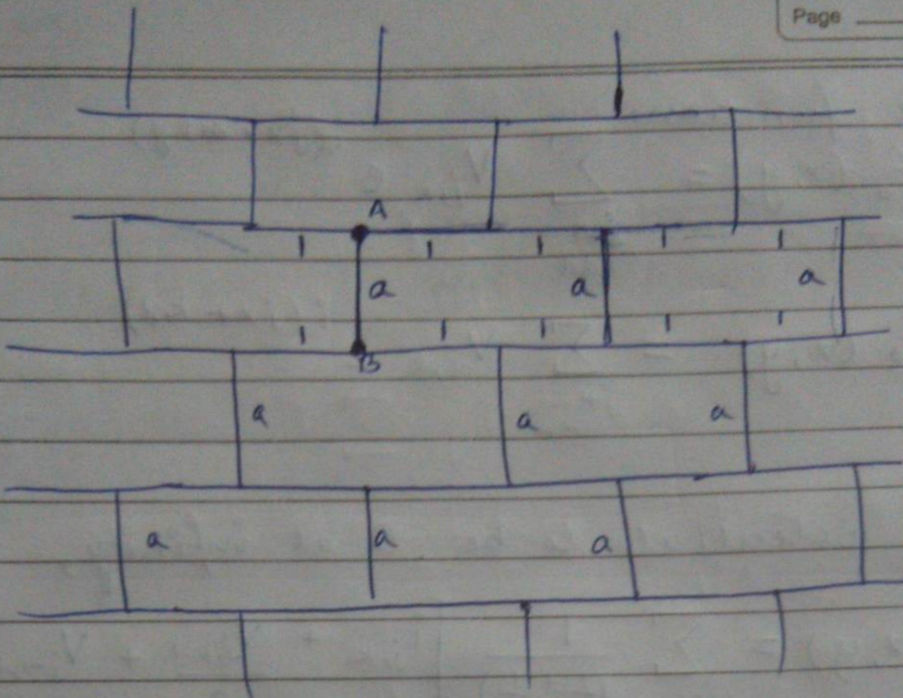
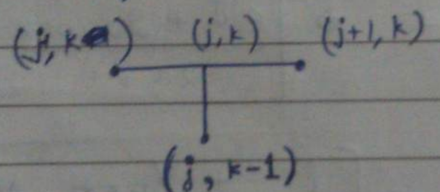


$a = 2018$

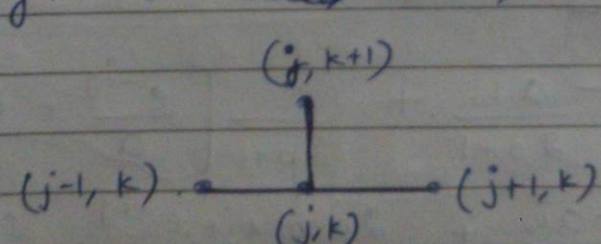
Assign cartesian coordinates to all points (nodes), such that

$A \equiv (0,1) \quad B \equiv (0,0)$.

Note that if a point (j,k) has $j+k$ odd, then the junction at that point looks like



If $j+k$ is even, the junction is like



Assign potentials $V(j,k) \equiv V_{j,k}$ to points (j,k), $(j,k) \in \mathbb{Z}^2$

Also let $i_{j,k}$ be the current going into the junction at node (j,k).

By Ohm's law, if $j+k$ is odd,

$$i_{j,k} = \frac{V_{j,k} - V_{j,k-1}}{a} + 2V_{j,k} - V_{j-1,k} - V_{j+1,k}$$

For $j+k$ even, $$i_{j,k} = \frac{V_{j,k} - V_{j,k+1}}{a} + 2V_{j,k} - V_{j-1,k} - V_{j+1,k}.$$

Define functions

$$f_o(x,y) = \sum_{\substack{j+k \\ \text{odd}}} V_{j,k}\, e^{i(jx+ky)}$$

$$f_e(x,y) = \sum_{\substack{j+k \\ \text{even}}} V_{j,k}\, e^{i(jx+ky)}$$

Taking potentials to be 0 at infinity,

$$f_o(x,y) = \sum_{\substack{j+k \\ \text{odd}}} \frac{1}{\left(2+\frac{1}{a}\right)} \left[i_{j,k} + \frac{V_{j,k-1} + V_{j-1,k} + V_{j+1,k}}{a} \right] \cdot e^{i(jx+ky)}$$

$$= \frac{1}{\left(2+\frac{1}{a}\right)} i_o e^{iy} + \frac{1}{\left(2+\frac{1}{a}\right)} \cdot \left[\frac{e^{iy} + e^{ix} + e^{-ix}}{a} \right] f_e(x,y)$$

$$= \frac{a i_o e^{iy}}{2a+1} + \left[\frac{e^{iy} + 2a\cos x}{2a+1} \right] f_e(x,y)$$

Also,

$$f_e(x,y) = \sum_{\substack{j+k \\ \text{even}}} \frac{1}{2+\frac{1}{a}} \left[i_{j,k} + \frac{V_{j,k+1} + V_{j-1,k} + V_{j+1,k}}{a} \right] \cdot e^{i(jx+ky)}$$

$$= \frac{-a i_o}{2a+1} + \frac{a}{2a+1} \left[\frac{e^{-iy}}{a} + 2\cos x \right] f_o(x,y)$$

$$= \frac{-a i_o}{2a+1} + \frac{2a\cos x + e^{-iy}}{2a+1} \cdot f_o(x,y)$$

$$\Rightarrow f_o(x,y) = \frac{a i_o e^{iy}}{2a+1} + \frac{(e^{iy} + 2a\cos x)(-a i_o)}{(2a+1)^2} + \frac{(e^{iy} + 2a\cos x)(e^{-iy} + 2a\cos x)}{(2a+1)^2} \cdot f_o(x,y)$$

$$\Rightarrow \left[4a^2 + 4a + 1 - 1 - 4a\cos x\cos y - 4a^2\cos^2 x \right] f_o(x,y)$$

$$= a i_o e^{iy}(2a+1) - a i_o e^{iy} - 2a^2 i_o \cos x$$

$$= 2a^2 i_o \left(e^{iy} - \cos x \right),$$

$$\Rightarrow f_o(x,y) = \frac{a\,\cancel{2a^2}\, i_o\,(e^{iy} - \cos x)}{2\,\cancel{4a}\,[a\sin^2 x + 1 - \cos x \cos y]}$$

$$\boxed{f_o(x,y) = \frac{a i_o\,(e^{iy} - \cos x)}{2[a\sin^2 x + 1 - \cos x\cos y]}}$$

$$f_e(x,y) = \frac{-a i_o}{2a+1} + \frac{(2a\cos x + e^{-iy})}{2a+1}\,\frac{a i_o\,(e^{iy} - \cos x)}{2(a\sin^2 x + 1 - \cos x\cos y)}$$

$$= \frac{a i_o\left[-2a\sin^2 x - 2 + 2\cos x\cos y + 1 - 2a\cos^2 x + 2a\cos x\, e^{iy} - \cos x\, e^{-iy}\right)}{2(2a+1)(a\sin^2 x + 1 - \cos x\cos y)}$$

$$= \frac{a i_o\,(-2a - 1 + 2a\cos x\cos y + 2a\cos x\sin y\, i + 2\cos x\cos y - \cos x\cos y + \cos x\sin y\, i)}{2(2a+1)(a\sin^2 x + 1 - \cos x\cos y)}$$

$$= \frac{a i_o\,(-2a-1 + (2a+1)\cos x\cos y + (2a+1)\cos x\sin y\, i)}{2(2a+1)(a\sin^2 x + 1 - \cos x\cos y)}$$

$$\boxed{f_e(x,y) = \frac{a i_o\,(-1 + \cos x\, e^{iy})}{2(a\sin^2 x + 1 - \cos x\cos y)}}$$

By def^n,

$$f_o(x,y) = \sum_{j+k \text{ odd}} V_{j,k}\, e^{i(jx+ky)}$$

$$\Rightarrow f_o(x,y) = V_{0,1}\, e^{iy} + \sum_{\substack{j+k \text{ odd} \\ (j,k)\neq(0,1)}} V_{j,k}\, e^{i(jx+ky)}$$

$$\Rightarrow f_o(x,y)\, e^{-iy} = V_{0,1} + \sum_{\substack{j+k \text{ odd} \\ (j,k)\neq(0,1)}} V_{j,k}\, e^{i(jx+(k-1)y)}$$

Now we want to isolate just $V_{0,1}$ from this sum.

$$\int_0^{2\pi}\int_0^{2\pi} f_o(x,y)\, e^{-iy}\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^{2\pi}\Big[V_{0,1} + \sum_{\substack{j+k \text{ odd} \\ j,k \neq (0,1)}} V_{j,k}\cdot e^{i(jx+(k-1)y)}\Big] dy\, dx$$

$$= 4\pi^2 V_{0,1} + \int_0^{2\pi}\int_0^{2\pi} \sum_{\substack{j+k \text{ odd} \\ (j,k)\neq(0,1)}} V_{j,k}\, e^{i(jx+(k-1)y)}\, dy\, dx$$

$$= 4\pi^2 V_{0,1}.$$

Similarly $\displaystyle\int_0^{2\pi}\int_0^{2\pi} f_e(x,y)\, dy\, dx = 4\pi^2 V_{0,0}.$

$$-4\pi^2 (V_{0,0} - V_{0,1}) = 4\pi^2 iR.$$

$$\Rightarrow -4\pi^2 R = \int_0^{2\pi}\int_0^{2\pi} \frac{a\,[-1+\cos x\, e^{iy} - 1 + \cos x\, e^{-iy}]}{2(a\sin^2 x + 1 - \cos x\cos y)}\, dy\, dx$$

$$\Rightarrow -R = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{a\,[2\cos x\cos y - 2]}{2(a\sin^2 x + 1 - \cos x\cos y)}\, dy\, dx$$

$$\Rightarrow -R = \frac{a}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{(\cos x\cos y - 1)}{a\sin^2 x + 1 - \cos x\cos y}\, dy\, dx$$

$\boxed{\text{Now we find the integral.}}$

$$\int_0^{2\pi}\int_0^{2\pi} \frac{a\sin^2 x - (a\sin^2 x + 1 - \cos x\cos y)}{a\sin^2 x + 1 - \cos x\cos y}\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^{2\pi}\left[\frac{a\sin^2 x}{a\sin^2 x + 1 - \cos x\cos y} - 1\right] dy\, dx$$

$$= -4\pi^2 + \int_0^{2\pi}\int_0^{2\pi} \frac{a\sin^2 x}{a\sin^2 x + 1 - \cos x\cos y}\, dy\, dx$$

Note that the integral is symmetric about $x=\pi$, & $y=\pi$, so it equals

$$-4\pi^2 + 4\int_0^{\pi}\int_0^{\pi} \frac{a\sin^2 x}{a\sin^2 x + 1 - \cos x\cos y}\,dy\,dx$$

Now we look at 2 cases, so break this integral into 2 parts

$$-4\pi^2 + 4\int_0^{\pi/2}\int_0^{\pi} \frac{a\sin^2 x\,dy\,dx}{a\sin^2 x + 1 - \cos x\cos y} + 4\int_{\pi/2}^{\pi}\int_0^{\pi} \frac{a\sin^2 x\,dy\,dx}{a\sin^2 x + 1 - \cos x\cos y}$$

$= \cancel{-4\pi}$

$= \cancel{-4\pi^2 + 4I_1 + 4I_2}$.

For $x \in (0, \pi/2)$, $\dfrac{a\sin^2 x + 1}{\cos x} > 0$

$$\Rightarrow \int_0^{\pi} \frac{a\sin^2 x}{a\sin^2 x + 1 - \cos x\cos y}\,dy$$

$$= \frac{a\sin^2 x}{\cos x}\int_0^{\pi} \frac{1}{\frac{a\sin^2 x + 1}{\cos x} - \cos y}\,dy$$

$$\frac{a\sin^2 x + 1}{\cos x} = \alpha > 1, \quad \& \quad \tan\frac{y}{2} = t$$

$\Rightarrow$ it equals

$$\frac{a\sin^2 x}{\cos x}\left\{\int_0^{\infty} \frac{1}{\alpha - \frac{1-t^2}{1+t^2}}\left(\frac{2dt}{1+t^2}\right)\right\} \cancel{+ \int_{-\infty}^{0} \frac{1}{\alpha - \left(\frac{1-t^2}{1+t^2}\right)}\frac{2dt}{1+t^2}}$$

$$= \frac{a\sin^2 x}{\cos x}\int_0^{\infty} \frac{2\,dt}{(\alpha-1) + (\alpha+1)t^2}$$

$$= \frac{2a\sin^2 x}{\cos x\,(\alpha+1)}\cdot\frac{1}{\sqrt{\frac{\alpha-1}{\alpha+1}}}\left[\tan^{-1}\left(\frac{t}{\sqrt{\frac{\alpha-1}{\alpha+1}}}\right)\right]_0^{\infty}$$

$$= \frac{a\sin^2 x}{\cos x\sqrt{\alpha^2-1}}\,\pi = \frac{a\pi\sin^2 x}{\sqrt{(1+a\sin^2 x)^2 - \cos^2 x}}$$

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For $x \in \left(\frac{\pi}{2}, \pi\right)$, $\frac{a\sin^2 x+1}{\cos x} < -1$, let $\alpha_0 = \frac{a\sin^2 x+1}{-\cos x} > 1$

$$\Rightarrow \int_0^{\pi} \frac{a\sin^2 x}{a\sin^2 x+1-\cos x\cos y}\,dy$$

$$= \frac{a\sin^2 x}{-\cos x}\int_0^{\infty} \frac{1}{\alpha_0+\left(\frac{1-t^2}{1+t^2}\right)}\frac{2\,dt}{1+t^2}$$

$$= \frac{2a\sin^2 x}{-\cos x}\int_0^{\infty} \frac{dt}{(\alpha_0+1)+(\alpha_0-1)t^2}$$

$$= \frac{2a\sin^2 x}{-\cos x\sqrt{\alpha_0^2-1}}\tan^{-1}\left[\frac{t}{\sqrt{\frac{\alpha_0+1}{\alpha_0-1}}}\right]_0^{\infty}$$

$$= \frac{a\sin^2 x}{-\cos x\sqrt{\alpha_0^2-1}}\pi$$

$$= \frac{a\sin^2 x\,\pi}{\sqrt{(1+a\sin^2 x)^2-\cos^2 x}} \quad (\text{as } -\cos x > 0)$$

$$\Rightarrow R = \frac{a}{4\pi^2}\left[-4\pi^2+4\int_0^{\pi}\frac{a\pi\sin^2 x}{\sqrt{(1+a\sin^2 x)^2-\cos^2 x}}\,dx\right]$$

$$= \frac{a}{4\pi^2}\left[-4\pi^2+4a\pi\times\int_0^{\pi}\frac{\sin^2 x}{\sqrt{(1+a\sin^2 x)^2-\cos^2 x}}\,dx\right]$$

$$\frac{\tan^{-1}\left(\frac{2018}{\sqrt{4037}}\right)}{1009} = 0.0015256$$

$$\Rightarrow \boxed{R = 40.4358}$$

Calculation of $\displaystyle\int_0^{\pi} \frac{\sin^2 x\, dx}{\sqrt{(1+a\sin^2 x)^2-\cos^2 x}}$

$$\int_0^{\pi} \frac{\sin^2 x}{\sqrt{(1+a\sin^2 x)^2-1+\sin^2 x}}\,dx$$

$$=\int_0^{\pi} \frac{\sin x}{\sqrt{a(2+a\sin^2 x)+1}}\,dx$$

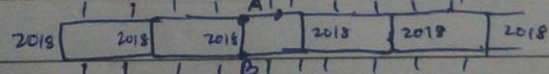
$$=\int_0^{\pi} \frac{-d(\cos x)}{\sqrt{2a+a^2+1-a^2\cos^2 x}}$$

$$=\frac{1}{a}\int_{-1}^{1} \frac{dt}{\sqrt{\left(\frac{a+1}{a}\right)^2-t^2}}$$

$$=\frac{1}{a}\times 2\sin^{-1}\left(\frac{a}{a+1}\right)$$

$$=\frac{2}{2018}\sin^{-1}\left(\frac{2018}{2019}\right)=\frac{\tan^{-1}\left(\frac{2018}{\sqrt{4037}}\right)}{1009}$$

It is worthwhile to remark that an upper bound on the resistance found by removing all resistances except the ones on a strip containing A & B, as shown



gives a bound of $\frac{2018}{\sqrt{2019}}$, i.e. 44.9 which is remarkably close to the exact answer $\frac{-2018^2}{\pi}\cdot\frac{\tan^{-1}\left(\frac{2018}{\sqrt{4037}}\right)}{1009}+2018$

$$\approx 40.4358$$