Solution for Problem 1

Satoshi Yoshida

If some wires are added to the circuit in parallel, the resistance between A and B becomes lower.

Rough explanation for underlined statement: The current can go through not only the original path but also the added path and more current can flow at the same voltage.

More mathematical explanation for underlined statement: Let $k = 1, 2, \ldots$ be the number for each path in the circuit, I_k be the current that go through the k-th path, $J(I_1, I_2, \ldots)$ be the constraints (the sum of the current flowing into a point is $-I$ (node B), I (node A), 0 (otherwise)) and $\Omega \coloneqq \{(I_k) | I_k \in \mathbb{R},\ J(I_1, I_2, \ldots)\}$. If we input the current I at node A and output I at node B, (I_k) that minimizes the Joule heat is chosen from Ω. Therefore, if we add some paths to the circuit in parallel, Ω becomes larger and the Joule heat becomes smaller. The Joule heat is ρI^2 so the residence between A and B becomes smaller.

Inversely, if some of the wires are removed from the circuit, the residence between A and B becomes larger.

Then, let's think about the circuits 1 (residence R) and 2 (residence r) shown below (Fig 1 and 2).

Using the underlined statement shown above, $r \leq \rho \leq R$.

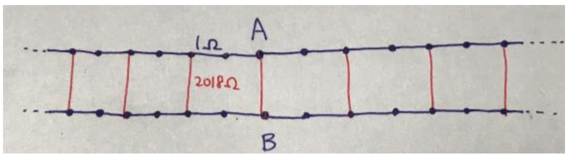

Fig 1. circuit 1

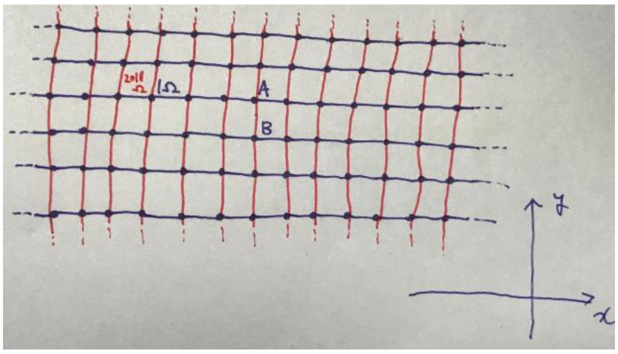

Fig 2. circuit 2

1. Circuit 1

The circuit 1 can be divided to three parts, two parts shown in Fig 3 and one wire whose residence is $2018\,\Omega$. Let R_n be shown in Fig 3. Then,

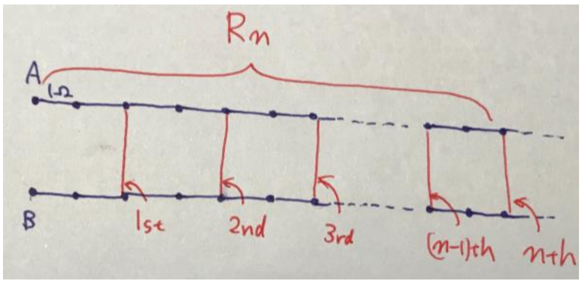

Fig 3. Infinite ladder circuit

$$2+\frac{1}{\frac{1}{R_n/\Omega}+\frac{1}{2018}}+2=R_{n+1}/\Omega$$

(see Fig 4)

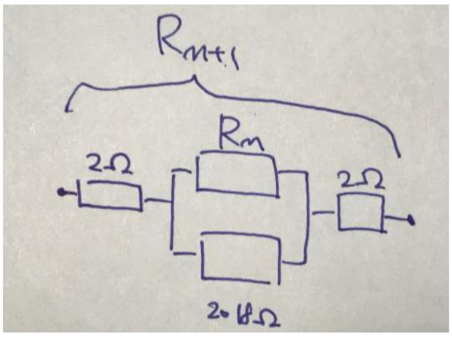

Fig 4. Idea to calculate $\boldsymbol{R_n}$

Let $R_\infty=\lim_{n\to\infty}R_n=\lim_{n\to\infty}R_{n+1}$. Then,

$$2+\frac{1}{\frac{1}{R_\infty/\Omega}+\frac{1}{2018}}+2=R_\infty/\Omega$$

$$R_\infty=2+2\sqrt{2019}\,\Omega$$

In circuit 1, two R_∞s and $2018\,\Omega$ are connected at A and B in parallel, so the residence between A and B is

$$R=\frac{1}{\frac{1}{2+2\sqrt{2019}}+\frac{1}{2+2\sqrt{2019}}+\frac{1}{2018}}\,\Omega=\frac{2018}{\sqrt{2019}}\,\Omega=44.91\ldots\,\Omega<44.92\,\Omega$$

2. Circuit 2

Think about the situation that the current I is input at node A and no current is input or output at any node other than A (including B). And let (x, y) be the coordinate of each node. ($(0, 0)$ is node A and $(0, -1)$ is node B). Let $V(x, y)$ be the electrical potential at (x, y) and $I(x, y)$ be the current flowing out of (x, y). Kirchhoff's second law shows that

$$I(x,y) = \frac{2V(x,y) - V(x-1,y) - V(x+1,y)}{1\,\Omega} + \frac{2V(x,y) - V(x,y-1) - V(x,y+1)}{2018\,\Omega} \cdots\cdots (1)$$

To solve this, assume that

$$V(x,y) = \int_0^{2\pi} d\beta\, F(\beta) e^{i|x|\alpha(\beta) + iy\beta} \cdots\cdots (2)$$

where $F(\beta), \alpha(\beta)$ are the functions of β.

If $x \neq 0$, $I(x, y) = 0$. Then,

$$\int_0^{2\pi} d\beta\, F(\beta) e^{i|x|\alpha(\beta) + iy\beta} \left(2 - 2\cos\alpha(\beta) + \frac{2 - 2\cos\beta}{2018}\right) = 0$$

Therefore,

$$2 - 2\cos\alpha(\beta) + \frac{2 - 2\cos\beta}{2018} = 0$$

$$e^{i\alpha(\beta)} = 1 + \frac{1}{2018}(1 - \cos\beta) - \frac{1}{2018}\sqrt{(1-\cos\beta)(4037 - \cos\beta)} \cdots\cdots (3)$$

If $x = 0$,

$$\int_0^{2\pi} d\beta\, F(\beta) e^{iy\beta} \left(2 - 2e^{i\alpha(\beta)} + \frac{2 - 2\cos\beta}{2018}\right) = I(0, y)$$

Using the theorem of Fourier analysis, this is equivalent with this:

$$F(\beta)\left(2 - 2e^{i\alpha(\beta)} + \frac{2 - 2\cos\beta}{2018}\right) = \frac{1}{2\pi}\sum_{x=\infty}^{\infty} I(0, y) = \frac{I}{2\pi}$$

By (3), $2 - 2e^{i\alpha(\beta)} + \frac{2-2\cos\beta}{2018} = -2i\sin\alpha(\beta) = \frac{2}{2018}\sqrt{(1-\cos\beta)(4037-\cos\beta)}$.

Then,

$$F(\beta) = \frac{2018 I}{4\pi\sqrt{(1-\cos\beta)(4037-\cos\beta)}} \cdots\cdots (4)$$

If we define $F(\beta), \alpha(\beta)$ as (3) and (4), (1) is true. Therefore, the assumption (2) is consistent with (1).

Then, the difference between the electrical potential at A and that at B is

$$V(0,0) - V(0,-1) = I \int_0^{2\pi} d\beta \frac{2018\left(1 - e^{i\beta}\right)}{4\pi\sqrt{(1-\cos\beta)(4037-\cos\beta)}}$$

Similarly, if the current I is output at B and no current is input or output at any node other than B, the difference between the electrical potential at A and that at B is the same as shown above. Using the principle of superposition, we know that if the current I is input and output at A and B, respectively, the electrical potential at A and that at B is twice as much as shown above. Therefore, the residence between A and B of the circuit 2 is

$$r = 2\int_0^{2\pi} d\beta \frac{2018\left(1 - e^{i\beta}\right)}{4\pi\sqrt{(1-\cos\beta)(4037-\cos\beta)}}\ \Omega = 28.59 \ldots \Omega > 28.59\ \Omega$$

I used Wolfram Alpha to calculate this.

$$\frac{R}{r} = 1.57 \ldots \leq 2$$

Then, the answer is

$$r = 28.59\ \Omega$$

$$R = 44.92\ \Omega$$