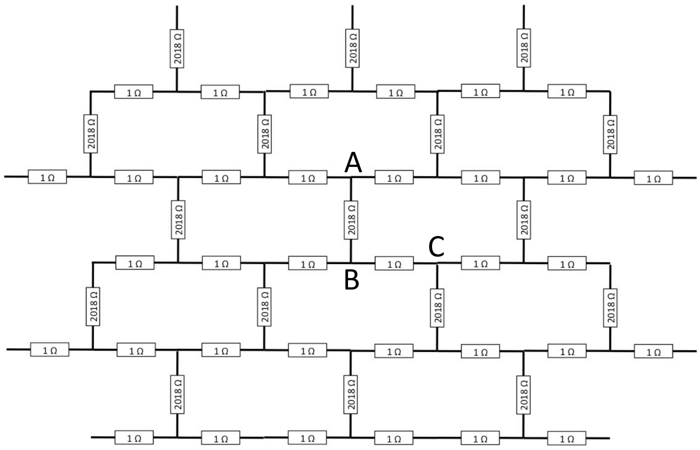
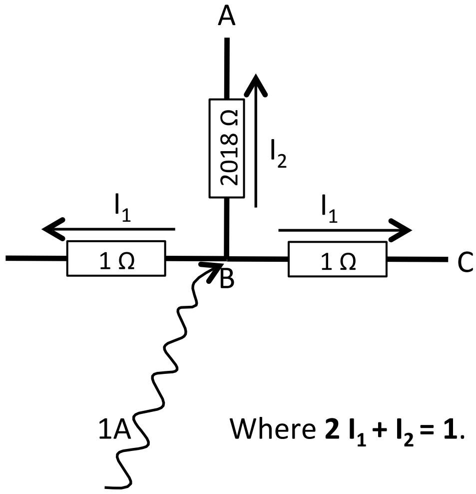
Physics Cup Problem 1





Injecting 1A current into node B, we have (by symmetry) the following distribution of current:

Where $\mathbf{2\, I_1 + I_2 = 1}$.

Similarly, if 1A is withdrawn from node B, the current directions will be reversed.

Consider the following scenario: 1A current is injected into node B, and 1A current is withdrawn from node A. By **Principle of Superposition**, one has that the current flow through the 2018 Ω resistor between A and B is $\mathbf{2\, I_2}$ (from B to A). The potential difference between B and A is thus V_{AB} = (2018 Ω)(2 I_2) = 4036 I_2 and current flow is I_{AB} = 1A.

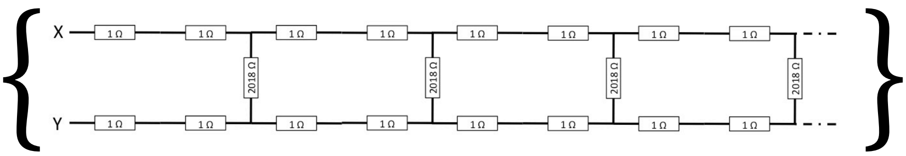
Thus $R_{AB} = V_{AB} / I_{AB} = \mathbf{4036\, I_2}$.

Similarly, consider: 1A current is injected into node B, and 1A current is withdrawn from node C. By Principle of Superposition, current flow through the 1 Ω resistor between B and C is 2 I_1 (from B to C). Thus V_{BC} = (1 Ω)(2 I_2) = 2 I_2 and I_{BC} = 1A.

Thus $R_{BC} = V_{BC} / I_{BC} = \mathbf{2\, I_1}$.

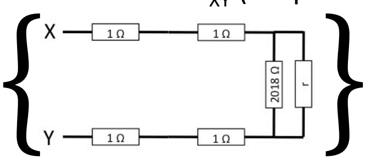
2 I_1 + I_2 = 1, thus $\mathbf{R_{AB} / 4036 + R_{BC} = 1}$.

Let r =



Where {circuit} denotes the effective resistance R_{XY} (for points X, Y marked on the circuit).

Then by simple recurrence, one has r =

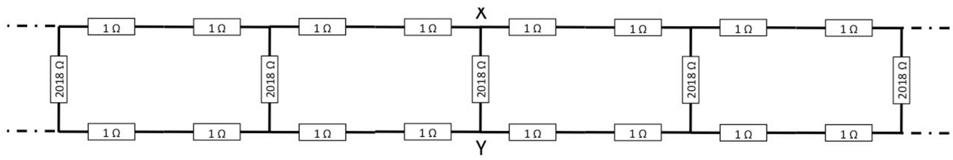


Thus r = 1 + 1 + (2018 // r) + 1 + 1 which simplifies to $\mathbf{r^2 = 4r + 8072}$.

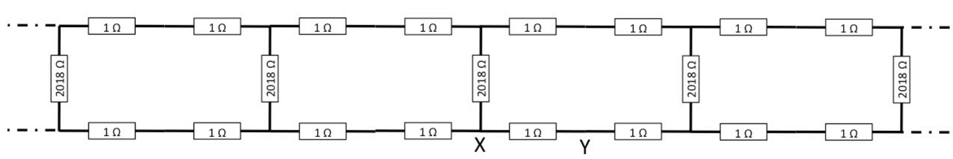
Solving the quadratic, one arrives at $r = 2 \pm 2\sqrt{(2019)}$. Rejecting the negative solution, one has:

$$\mathbf{r = 2 + 2\sqrt{(2019)}}$$

Given any resistor circuit, the removal of any nodes or connections will certainly **not decrease** the effective resistance between any pair of points. By removing all nodes and resistor connections from the infinite lattice apart from the horizontal ladder rung containing points A, B and C, we may thus **find upper bounds for R_{AB} and R_{BC}**:

$R_{AB} \leq$ {  }

Thus $R_{AB} \leq 2018 \,//\, r \,//\, r$
$= 2018r \div (r + 4036)$
$=$ **$2018 \div \sqrt{2019}$.** 🡐 sub. $r = 2 + 2\sqrt{2019}$

$R_{BC} \leq$ {  }

Thus $R_{BC} \leq 1 \,//\, [(2018 \,//\, r) + (r - 1)]$
$= (r^2 + 4035r - 2018) \div (r^2 + 4036r)$.

An upper bound for R_{BC} gives a lower bound for R_{AB}:

$R_{AB} = 4036\,(1 - R_{BC}) \geq 4036\,[1 - (r^2 + 4035r - 2018) \div (r^2 + 4036r)]$
$= 4036\,(r + 2018) \div (r^2 + 4036r)$
$=$ **$1009 \div \sqrt{2019}$.** 🡐 sub. $r = 2 + 2\sqrt{2019}$

Thus: **$1009 \div \sqrt{2019}\ \Omega \leq \rho = R_{AB} \leq 2018 \div \sqrt{2019}\ \Omega$**
where the upper bound is exactly twice the lower bound.