Physics Cup Problem 1

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We shall try to find the exact value of the resistance between the 2 points, for the general case where the 2018 Ω resistor is replaced by r

Let us choose a cartesian coordinate system with point A as the origin (thus point B is (0,-1) and the points adjacent to A are (±1,0)).

Assign to the point (i,j) it's potential V_{ij} and let the current being taken out of the point be I_{ij}. Writing kirchoff's node law at the point,

$$2V_{ij} - V_{i-1j} - V_{i+1j} + \frac{V_{ij} - V_{ij\pm1}}{r} = I_{ij}$$

$$\Longrightarrow V_{ij} = \frac{rI_{ij} + r(V_{i+1j} + V_{i-1j}) + V_{ij\pm1}}{2r+1}$$

Here, whether we choose the plus or minus in $V_{ij\pm1}$ depends on whether $i + j$ is odd (plus) or even (minus).

Let us calculae the fourier series with the potential as coefficients (o is the root of -1)

$$F_1(x,y) = \sum_{i+jeven} V_{ij}e^{o(ix+jy)}$$

$$= \sum \frac{rI_{ij}e^{o(ix+jy)} + r(V_{i+1j}e^{o(ix+jy)} + V_{i+1j}e^{o(ix+jy)}) + V_{ij+1}e^{o(ix+jy)}}{2r+1}$$

$$= \frac{rI + \sum_{i+jodd} V_{ij}e^{o(ix+jy)}(2r\cos(x) + e^{oy})}{2r+1}$$

$$= \frac{rI + F_2(x,y)(2r\cos(x) + e^{oy})}{2r+1}$$

Where we used the fact that $I_{0,-1} = I$(current being removed) and F_2 is defined by:

$$F_2(x,y) = \sum_{i+jodd} V_{ij} e^{o(ix+jy)}$$

Using similar line of reasoning above, we get

$$F_2 = \frac{-Ire^{-oy} + F_1(2r\cos(x) + e^{-oy})}{2r+1}$$

Solving the 2 equations, we get

$$F_1 = \frac{Ir}{2} \frac{1 - \cos(x)e^{-oy}}{r\sin^2(x) - \cos(x)\sin(x) + 1}$$

$$F_2 = \frac{Ir}{2} \frac{\cos(x) - e^{-oy}}{r\sin^2(x) - \cos(x)\sin(x) + 1}$$

Thus we can find the resistance using $\rho = \frac{V_{00} - V_{0,-1}}{I}$, which comes out to be:

$$\rho = \frac{r}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\cos(x)\cos(y) - 1}{r\sin^2(x) - \cos(x)\sin(x) + 1} dxdy$$

Using OCTAVE, we get $\rho = 40.436$ when $r = 2018$.

ANSWER

we can choose r=30 and R=60 and $r < \rho < R$.