Physics Cup - Problem 2

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First of all, we are going to determine the orbit of the spaceship. We introduce a reference frame where every movement happens in the x-direction and at $t = 0$ the spaceship is at $x = 0$ and moves with speed $u = 0$. (Note that this is an inertial reference frame.)
By the laws of special relativity, the acceleration of the spaceship a perceived by an observer in our reference frame is given by:

$$a = g\left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}} \tag{1}$$

where u is the current speed of the spaceship and c is the speed of light. Plugging in the definition of a:

$$\frac{du}{dt} = g\left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}}$$

Now we solve this differential equation:

$$\frac{du}{\left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}}} = gdt$$

Let's integrate both sides:

$$\int_0^u \frac{du'}{\left(1 - \frac{u'^2}{c^2}\right)^{\frac{3}{2}}} = \int_0^t gdt'$$

To integrate the left-hand side, use the substitution $x = \frac{u}{c}$:

$$\int_0^x \frac{cdx'}{(1 - x'^2)^{\frac{3}{2}}} = gt$$

Another substitution: let $x = \sin\alpha$:

$$\int_0^\alpha \frac{c \cdot \cos(\alpha')d\alpha'}{(1 - \sin(\alpha')^2)^{\frac{3}{2}}} = gt$$

$$\int_0^\alpha \frac{c \cdot \cos(\alpha')d\alpha'}{\cos(\alpha')^3} = gt$$

$$\int_0^\alpha \frac{cd\alpha'}{\cos(\alpha')^2} = gt$$

$$c \cdot (\tan(\alpha) - \tan(0)) = gt$$

So we get:

$$c \tan \alpha = gt$$

We know that $x = \sin \alpha$, thus:

$$c\sqrt{\frac{x^2}{1-x^2}} = gt$$

$$c\sqrt{\frac{\frac{u^2}{c^2}}{1-\frac{u^2}{c^2}}} = gt$$

$$\sqrt{\frac{c^2u^2}{c^2-u^2}} = gt$$

After rearranging, this yields:

$$u = \frac{cgt}{\sqrt{c^2+g^2t^2}} \tag{2}$$

Plugging in the definition of u, we get another differential equation:

$$\frac{dx}{dt} = \frac{cgt}{\sqrt{c^2+g^2t^2}}$$

$$dx = \frac{cgt \cdot dt}{\sqrt{c^2+g^2t^2}}$$

Integrating both sides, with substitution $y = \sqrt{c^2+g^2t^2}$:

$$dy = \frac{2g^2tdt}{2\sqrt{c^2+g^2t^2}} = \frac{g^2tdt}{\sqrt{c^2+g^2t^2}}$$

$$x = \int_c^y \frac{c}{g}dy' = \frac{c}{g}(y-c) = \frac{c}{g}\left(\sqrt{c^2+g^2t^2}-c\right)$$

This means:

$$x = \frac{c}{g}\left(\sqrt{c^2+g^2t^2}-c\right) \tag{3}$$

Let us launch a missile at $t = 0$ with speed v. Now we are going to calculate the time t required to catch the missile in our reference frame. As at $t = 0$ the spaceship is at rest, this means that in the inertial frame the speed of the missile is simply v. At time t, the spaceship catches the missile, so:

$$x(t) = vt$$

$$\frac{c}{g}\left(\sqrt{c^2+g^2t^2}-c\right) = vt$$

$$\sqrt{c^2+g^2t^2} = \frac{gvt}{c} + c$$

$$c^2+g^2t^2 = \frac{g^2v^2t^2}{c^2} + 2gvt + c^2$$

$$\left(g^2 - \frac{g^2v^2}{c^2}\right)t^2 - 2gvt = 0$$

$$t \cdot \left(g^2 t - \frac{g^2v^2}{c^2}t - 2gv\right) = 0$$

$t = 0$ corresponds to the launch of the missile, so we need the other root:

$$t = \frac{2gv}{g^2 - \frac{g^2v^2}{c^2}} = \frac{2vc^2}{g(c^2 - v^2)} \tag{4}$$

As a final step, we are going to calculate the proper time τ elapsed in the spaceship. For infinitesimally small changes:

$$d\tau = dt\sqrt{1 - \frac{u^2}{c^2}}$$

Integrating both sides, using (2):

$$\tau = \int_0^t dt\sqrt{1 - \frac{g^2t^2}{c^2 + g^2t^2}} = \int_0^t \frac{c \cdot dt}{\sqrt{c^2 + g^2t^2}}$$

Using the substitution $y = \frac{gt}{c}$:

$$\tau = \int_0^y \frac{c}{g} \cdot \frac{dy'}{\sqrt{1 + y'^2}}$$

This is a standard integral (here $\sinh^{-1}$ denotes the inverse hyperbolic sine):

$$\tau = \frac{c}{g}\left(\sinh^{-1}(y) - \sinh^{-1}(0)\right) = \frac{c}{g}\sinh^{-1}\left(\frac{gt}{c}\right)$$

Note that $\sinh^{-1}(x) = \ln(x + \sqrt{1 + x^2})$:

$$\tau = \frac{c}{g}\ln\left(\frac{gt}{c} + \sqrt{1 + \frac{g^2t^2}{c^2}}\right)$$

Using (4):

$$\tau = \frac{c}{g}\ln\left(\frac{2vc}{c^2 - v^2} + \sqrt{1 + \frac{4v^2c^2}{(c^2 - v^2)^2}}\right) = \frac{c}{g}\ln\left(\frac{2vc}{c^2 - v^2} + \sqrt{\frac{(c^2 - v^2)^2 + 4v^2c^2}{(c^2 - v^2)^2}}\right) =$$

$$= \frac{c}{g}\ln\left(\frac{2vc}{c^2 - v^2} + \frac{c^2 + v^2}{c^2 - v^2}\right) = \frac{c}{g}\ln\left(\frac{(c + v)^2}{c^2 - v^2}\right) = \frac{c}{g}\ln\left(\frac{c + v}{c - v}\right)$$

So the proper time elapsed between catching the first and the second missile:

$$\Delta\tau = \frac{c}{g}\left[\ln\left(\frac{c + 2v}{c - 2v}\right) - \ln\left(\frac{c + v}{c - v}\right)\right] = \frac{c}{g}\ln\left[\frac{(c + 2v)(c - v)}{(c - 2v)(c + v)}\right]$$