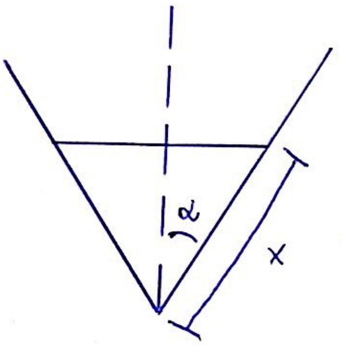
Physics Cup P4 Thomas Bergamaschi



As a function of x the center of mass height is

$$h_{cm} = -\left(l\cos\alpha + (l-x)\cos\alpha + \frac{x}{3}\cos\alpha\right) = -\left(2l - \frac{2x}{3}\right)\cos\alpha$$

And the total mass m is $m = \rho L l^2 \sin\alpha_0 \cos\alpha_0$

Where α_0 is the equilibrium angle, since m is constant:

$$m = \rho L x^2 \sin\alpha\cos\alpha \Rightarrow x^2 = \frac{m}{\rho L \sin\alpha\cos\alpha} = l^2 \frac{\sin\alpha_0\cos\alpha_0}{\sin\alpha\cos\alpha} = l^2 \frac{\sin 2\alpha_0}{\sin 2\alpha}$$

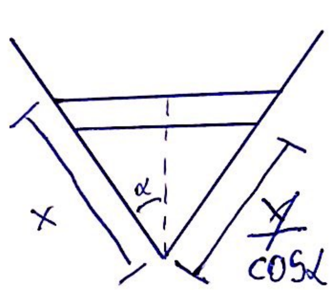
$\Rightarrow x = l\sqrt{\frac{\sin(2\alpha_0)}{\sin(2\alpha)}}$, Since potential energy is $U = mgh_{cm}$

$$\Rightarrow U = -mg\left(2l - \frac{2x}{3}\right)\cos\alpha = -2mgl\left(1 - \frac{1}{3}\sqrt{\frac{\sin(2\alpha_0)}{\sin(2\alpha)}}\right)\cos\alpha_0$$

At equilibrium $\frac{dU}{d\alpha} = 0 \Rightarrow$ Using Wolfram we find $\sin\alpha_0 = \frac{1}{\sqrt{6}}$, therefore

the angle between the plates is $2\alpha = 2\arcsin\left(\frac{1}{\sqrt{6}}\right) \simeq$

For the period of oscillation let us calculate the kinetic energy of the system using the figure below:



Drawing a set of horizontal lines we have that the distance between these lines is invariant, therefore

$$\frac{y}{x\cos\alpha} = C \Rightarrow \frac{y}{\cos\alpha}\sqrt{\sin 2\alpha} \Rightarrow \text{Therefore } y\sqrt{\tan\alpha} = C$$

$\hookrightarrow$ constant

$$\text{Since } y\sqrt{\tan\alpha} = C \Rightarrow \frac{d}{dt}\left(y\sqrt{\tan\alpha}\right) = 0 \Rightarrow \dot{y}\sqrt{\tan\alpha} + y\frac{d\sqrt{\tan\alpha}}{d(\tan\alpha)} \cdot \frac{\dot{\alpha}}{\cos^2\alpha} = 0$$

$$\Rightarrow \dot{y}\sqrt{\tan\alpha} + \frac{y}{2}\frac{\dot{\alpha}}{\sqrt{\tan\alpha}\cos^2\alpha} = 0 \Rightarrow \dot{y} = -\frac{\dot{\alpha}y}{2\sin\alpha\cos\alpha} = -\frac{\dot{\alpha}y}{\sin(2\alpha)}$$

Note that this velocity is relative to the bottom vertex, which has y coordinate $2l\cos\alpha$ and moves with velocity $V = 2l\dot{\alpha}\sin\alpha$

By conservation of volume $\vec{\nabla}\cdot\vec{v}=0$, where $\vec{v}$ is liquid velocity, where $\vec{v}=\dot{x}\hat{x}+\dot{y}\hat{y}$, this manner $\frac{\partial \dot{x}}{\partial x}+\frac{\partial \dot{y}}{\partial y}=0 \Rightarrow \frac{\partial \dot{x}}{\partial x}=\frac{\dot{\alpha}}{\sin(2\alpha)} \Rightarrow \dot{x}=\frac{\dot{\alpha}x}{\sin 2\alpha}$

For the kinetic energy associated to the x axis we have:

$$dT_x=\frac{\rho L dx dy}{2}\dot{x}^2 \Rightarrow T_x=\frac{\rho L}{2}\int_0^{l\cos\alpha}\int_{-y\tan\alpha}^{y\tan\alpha}\frac{\dot{\alpha}^2x^2}{\sin^2(2\alpha)}\,dx\,dy=\frac{\rho L\dot{\alpha}^2}{\cancel{2}\sin^2(2\alpha)}\int_0^{l\cos\alpha}\frac{\cancel{2}}{3}y^3\tan^3\alpha\,dy$$

$$\Rightarrow T_x=\frac{\rho L\dot{\alpha}^2}{3\sin^2(2\alpha)}\tan^3\alpha\cdot\frac{l^4\cos^4\alpha}{4}=\frac{\rho L\dot{\alpha}^2 l^4}{48\sin^2\alpha}\cdot\frac{\sin^3\alpha}{\cos^3\alpha}\cdot\frac{\cos^4\alpha}{\cos^2\alpha}=\frac{\rho L\dot{\alpha}^2 l^4}{48}\tan\alpha$$

For the y axis we have that the velocity with respect to the lab frame:

$$V_y=V-|\dot{y}|=\dot{\alpha}\left(2l\sin\alpha-\frac{y}{\sin(2\alpha)}\right), \text{ so } dT_y=\frac{\rho L dx dy}{2}V_y^2$$

$$\Rightarrow T_y=\frac{\rho L\dot{\alpha}^2}{2}\int_0^{l\cos\alpha}\int_{-y\tan\alpha}^{y\tan\alpha}\left(2l\sin\alpha-\frac{y}{\sin(2\alpha)}\right)^2dx\,dy=\frac{\rho L\dot{\alpha}^2}{\cancel{2}}\int_0^{l\cos\alpha}\cancel{2}y\tan\alpha\left(2l\sin\alpha-\frac{y}{\sin(2\alpha)}\right)^2dy$$

$$\Rightarrow T_y=\rho L\dot{\alpha}^2\tan\alpha\int_0^{l\cos\alpha}y\left(2l\sin\alpha-\frac{y}{\sin(2\alpha)}\right)^2dy=\rho L\dot{\alpha}^2\tan\alpha\cdot\frac{5l^4}{144}=\frac{\rho L l^4\dot{\alpha}^2\tan\alpha}{144}$$

So the total kinetic energy is $T=T_x+T_y=\rho L\dot{\alpha}^2l^4\tan\alpha\left(\frac{1}{48}+\frac{5}{144}\right)$

$$\Rightarrow T=\frac{\rho L\dot{\alpha}^2l^4\tan\alpha}{144}\cdot 8=\frac{\rho L l^4\dot{\alpha}^2\tan\alpha}{18}$$, since $\sin\alpha=\frac{1}{\sqrt{6}}$, $\cos\alpha=\sqrt{\frac{5}{6}}$, $\tan\alpha=\frac{1}{\sqrt{5}}$

$$T=\frac{\rho L l^4\dot{\alpha}^2\cancel{\tan\alpha}}{18\sqrt{5}}$$

Since the energy is $U(\alpha)=U(\alpha_0)+\underbrace{U'(\alpha_0)}_{0}(\alpha-\alpha_0)+\frac{U''(\alpha_0)(\alpha-\alpha_0)^2}{2}\simeq U(\alpha_0)+\frac{U''(\alpha_0)\Delta^2}{2}$

where $\Delta=\alpha-\alpha_0$, and $U=mg\,h_{cm}\Rightarrow U''=mg\,h_{cm}''$

Since $h_{cm}=-\left(2l-\frac{2x}{3}\right)\cos\alpha \Rightarrow h''=4l\sqrt{\frac{6}{5}} \Rightarrow U''=g\rho L l^2\sin\alpha\cos\alpha\cdot 4l\sqrt{\frac{6}{5}}$ $\cancel{\phi}$

$\Rightarrow$ Therefore $U(\alpha) = U(\alpha_0) + \dfrac{U'' \Delta^2}{2} = U(\alpha_0) + 2\rho g L l^3 \sin\alpha \cos\alpha \sqrt{\dfrac{6}{5}}\, \Delta^2$

So $\Omega^2 = 2\rho g \cancel{L} \cancel{l^3} \sin\alpha \cos\alpha \sqrt{\dfrac{6}{5}} \cdot \dfrac{1}{\dfrac{\cancel{\rho L l^4}}{18\sqrt{5}}} = \dfrac{g}{l} \cdot \sin\alpha \cos\alpha \sqrt{\dfrac{6}{5}} \cdot 18\sqrt{5} \cdot 2$

$\Rightarrow \Omega^2 = \dfrac{g}{l} \cdot \dfrac{1}{\sqrt{6}} \cdot \cancel{\sqrt{\dfrac{5}{6}}}\, \cancel{\sqrt{\dfrac{6}{5}}} \cdot 18 \cdot 2\sqrt{5} = \dfrac{g}{l} \cdot 36 \sqrt{\dfrac{5}{6}} \Rightarrow \Omega = \sqrt{\dfrac{g}{l}} \cdot 6 \sqrt{\sqrt{\dfrac{5}{6}}}$

$\Omega = 6\dfrac{g}{l} \cdot \sqrt{\sqrt{\dfrac{5}{6}}} = \dfrac{6g}{\sqrt{l}} \left(\dfrac{5}{6}\right)^{1/4} \Rightarrow 5{,}7327\sqrt{\dfrac{g}{l}}$

So $2\alpha = 2\arcsin\left(\dfrac{1}{\sqrt{6}}\right) \cong 48{,}1897$ and $\Omega = 5{,}7327\sqrt{\dfrac{g}{l}}$