Physics Cup 2018. Problem 5. CAI, Zixing, COMET, China

Aussume $\frac{kT}{mc^2} \ll 1$, so that higher order of relativity effects can be negleted.

(i). Caulcte the average force with velocity of $\vec{v}$.

In the rest frame, density of photons is n.

In the frame of the sphere, density of photons in $n \cdot \gamma$. $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

In a short time dt, total momentum recieved by the sphere is

$$d\bar{p}_s' = -\underbrace{n\gamma \cdot \pi R^2 c\, dt'}_{\text{numble of photons}} \cdot \int \underbrace{\frac{d\Omega}{4\pi}}_{\text{properbility of right direction}} \cdot \underbrace{\int f(\nu)\, d\nu}_{\text{properbility of right frequeney}} \cdot \frac{h\nu'}{c} \cdot \cos\theta'$$

where. $\nu' = \nu \cdot \gamma(1+\beta\cos\theta)$. (Doppler shift)

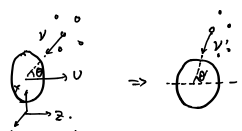
$$\cos\theta' = \frac{\cos\theta - \beta}{1-\beta\cos\theta}, \quad \beta = \frac{v}{c}.$$

Neglect higher order of $\frac{v}{c}$:

$$\frac{d\bar{p}_s}{dt} \simeq \frac{d\bar{p}_s'}{dt'} = -n \cdot \pi R^2 \cdot c \cdot \int_0^{\pi} d\theta \cdot \int_0^{\infty} h\nu f(\nu) d\nu \cdot \frac{2\pi\sin\theta}{4\pi} \cdot \frac{1}{c} \cdot (1+\beta\cos\theta)(\cos\theta - \beta\sin^2\theta)$$

(equal to the density of energy; zero; non zero)

$$\bar{F}_z = -\pi R^2 \cdot \frac{\pi k^4 T^4}{15 c^3 \hbar^3} \cdot \frac{1}{6} \cdot \frac{v}{c} = -k v$$



So we can assume total force to be:

$$\vec{F}(t) = -k\vec{v}(t) + \sum_n \vec{I}_n \delta(t - t_n)$$

where $\overline{\vec{I}_n} = 0$, $\overline{\vec{I}_m \cdot \vec{I}_n} = I^2 \delta_{mn}$

Newton's Law:

$$m\frac{d\vec{v}(t)}{dt} = \vec{F}(t) = -k\vec{v}(t) + \sum_n \vec{I}_n \delta(t - t_n)$$

Fourier tranformation:

$$\vec{v}(t) = \int_{-\infty}^{\infty} \tilde{\vec{v}}(\omega) e^{i\omega t} d\omega$$

$$\tilde{\vec{v}}(\omega) = \frac{1}{T}\int_0^T \vec{v}(t) e^{i\omega t} dt, \quad T \text{ is quiet a long time.}$$

$$i\omega m \tilde{\vec{v}}(\omega) = -k\tilde{\vec{v}}(\omega) + \frac{1}{T}\cdot\sum_n e^{-i\omega t_n}\vec{I}_n$$

$$\Rightarrow \tilde{\vec{v}}(\omega) = \frac{\frac{1}{T}\sum e^{-i\omega t_n}\vec{I}_n}{i\omega m + k}$$

Corelation: $t_1 = t_2 + t$

$$\overline{\vec{v}(t)\cdot\vec{v}(0)} = \overline{\vec{v}(t_1)\cdot\vec{v}(t_2)} = \overline{\int_{-\infty}^{\infty}\tilde{\vec{v}}(\omega_1) e^{i\omega_1(t_2+t)} d\omega_1 \int_{-\infty}^{\infty}\tilde{\vec{v}}(\omega_2) e^{i\omega_2 t_2} d\omega_2}$$

$$\overset{\text{average over } t_2}{=} \frac{1}{T}\cdot\int_0^T dt_2 \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2\, \overline{\tilde{\vec{v}}(\omega_1)\cdot\tilde{\vec{v}}(\omega_2)}\, e^{i(\omega_1+\omega_2)t_2} e^{i\omega_1 t} \quad [\tilde{\vec{v}}(\omega) = \tilde{\vec{v}}^*(-\omega)]$$

($\to \delta(\omega_1+\omega_2)$)

$$= \int_{-\infty}^{\infty} d\omega_1\, \tilde{\vec{v}}_1(\omega_1)\cdot\tilde{\vec{v}}(-\omega_1) e^{i\omega_1 t} = \int_{-\infty}^{\infty} d\omega\, \tilde{\vec{v}}^*(\omega)\cdot\tilde{\vec{v}}(\omega) e^{i\omega t}$$

$$\Rightarrow \overline{\vec{v}(t)\cdot\vec{v}(0)} = \int_{-\infty}^{\infty} \frac{\frac{1}{T}\sum_n e^{-i\omega t_n}\vec{I}_n}{i\omega m + k} \cdot \frac{\frac{1}{T}\sum_m e^{+i\omega t_m}\vec{I}_m}{-i\omega m + k} e^{i\omega t} d\omega$$

$$\overline{\vec{I}_n \cdot \vec{I}_m} = \delta_{nm} I_n^2$$

$$\Rightarrow \overline{\vec{v}(t)\cdot\vec{v}(0)} = \frac{\sum_n I_n^2}{T^2}\int_{-\infty}^{\infty}\frac{1}{k^2 + m^2\omega^2} e^{i\omega t} d\omega.$$

$$= \frac{\sum_n I_n^2}{T^2}\frac{1}{m^2}\cdot\frac{1}{2\frac{k}{m}}\cdot 2\pi\cdot\left(e^{-\frac{k}{m}t}\theta(t) + e^{\frac{k}{m}t}\theta(-t)\right)$$

$$\theta(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$\overline{\vec{v}(0)\cdot\vec{v}(0)} = \frac{\sum_n I_n^2}{T^2}\frac{1}{m^2}\cdot\frac{m}{k}\pi.$$

$$\Rightarrow \frac{\overline{\vec{v}(t)\cdot\vec{v}(0)}}{\overline{\vec{v}(0)\cdot\vec{v}(0)}} = e^{-\frac{k}{m}t}, \quad t > 0$$

The characteristic time of turning is

$$T = \frac{m}{k} = \frac{90\, m\, c^4 \cdot \hbar^3}{\pi^2 R^2 k^4 T^4}$$