Physics Cup Q2

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We fix S as the inertial rest frame of the spaceship when it ejects missiles (let this moment be $t = 0$). In frame S, the missiles are travelling at constant speeds v and $2v$ respectively; their motion with respect to time is thus:

$$x_1(t) = vt;\ x_2(t) = 2vt$$

The spaceship is undergoing proper acceleration g. Let S' be the inertial instantaneous rest frame of the spaceship at some time t; the Lorentz' transformation gives

$$\begin{cases} x' = \gamma(x - \beta ct) \\ ct' = \gamma(ct - \beta x) \end{cases}$$

thus

$$\frac{dx'}{dt} = \gamma\left(\frac{dx}{dt} - \beta c\right);\ \frac{d^2x'}{dt^2} = \gamma\frac{d^2x}{dt^2}$$

and time dilation $dt = \gamma dt'$ applies; thus

$$\frac{d^2x'}{dt'^2} = \left(\frac{dt}{dt'}\right)^2 \cdot \frac{d^2x'}{dt^2} = \gamma^3\frac{d^2x}{dt^2}$$

which gives the relation $\boldsymbol{a = g/\gamma^3}$ in frame S. We may solve this as follows:

$$\frac{dv}{dt} = g\left(1 - \frac{v^2}{c^2}\right)^{3/2}$$

Substituting $v = c \cdot \tanh\theta$:

$$\frac{dv}{d\theta}\cdot\frac{d\theta}{dt} = c \cdot \operatorname{sech}^2\theta \cdot \frac{d\theta}{dt} = g(1 - \tanh^2\theta)^{3/2} = g \cdot \operatorname{sech}^3\theta$$

$$\boldsymbol{\cosh\theta \cdot \frac{d\theta}{dt} = \frac{g}{c}}$$

Initial conditions $v_{t=0} = 0 \therefore \theta_{t=0} = 0$:

$$\int_0^\theta \cosh\theta\, d\theta = \sinh\theta = \int_0^t \frac{g}{c}\,dt = \frac{gt}{c}$$

$$v = c \cdot \tanh\sinh^{-1}\frac{gt}{c} = c \cdot \frac{gt/c}{\sqrt{(gt/c)^2 + 1}} = \frac{gt}{\sqrt{(gt/c)^2 + 1}}$$

$$x = \int_0^t v\,dt = \int_0^t \frac{gt}{\sqrt{(gt/c)^2 + 1}}\,dt = \frac{c^2}{g}\left[\sqrt{(gt/c)^2 + 1} - 1\right]$$

The missile catches up with the first and second missiles at $t = t_1$ and $t = t_2$ respectively (and corresponding to $\theta = \theta_1$ and $\theta = \theta_2$ respectively), which satisfy:

$$\frac{c^2}{g}\left[\sqrt{(gt_1/c)^2 + 1} - 1\right] = vt_1 \therefore c \cdot [\cosh\theta_1 - 1] = v \cdot \sinh\theta_1$$

$$\therefore 2c\sinh^2\frac{\theta_1}{2} = 2v\sinh\frac{\theta_1}{2}\cosh\frac{\theta_1}{2} \therefore \boldsymbol{\tanh\frac{\theta_1}{2} = \frac{v}{c}}$$

and similarly

$$\frac{c^2}{g}\left[\sqrt{(gt_2/c)^2 + 1} - 1\right] = 2vt_2 \therefore \boldsymbol{\tanh\frac{\theta_2}{2} = \frac{2v}{c}}$$

We want to find the proper time T between the two events:

$$T = \int dt' = \int_{t_1}^{t_2}\frac{dt}{\gamma} = \int_{t_1}^{t_2}\left(1 - \frac{v^2}{c^2}\right)^{1/2} dt = \int_{t_1}^{t_2}\operatorname{sech}\theta\,dt = \int_{\theta_1}^{\theta_2}\frac{c}{g}\,d\theta = \frac{c}{g}(\theta_2 - \theta_1)$$

The change of integration variable from t to θ makes use of $\cosh\theta \cdot \frac{d\theta}{dt} = \frac{g}{c}$ above. Thus:

$$T = \frac{2c}{g}\left(\tanh^{-1}\frac{2v}{c} - \tanh^{-1}\frac{v}{c}\right) = \frac{c}{g}\ln\left(\frac{1 + 2v/c}{1 - 2v/c}\cdot\frac{1 - v/c}{1 + v/c}\right) = \boldsymbol{\frac{c}{g}\ln\frac{(c + 2v)(c - v)}{(c - 2v)(c + v)}}$$