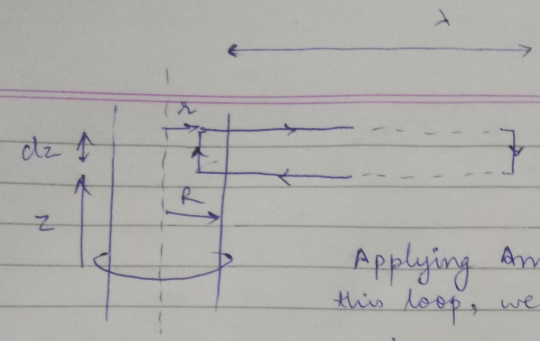
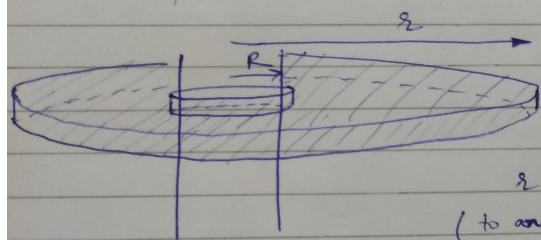


Applying Ampere's circulation theorem on this loop, we have

$$\frac{B_{ai}}{\mu} + \int_R^{\lambda} \frac{\partial B_{ro}}{\partial z}\, dr + \int_r^R \frac{\partial B_{ri}}{\mu\, \partial z}\, dr = 0$$

It is reasonable to approximate the last integral by 0 because the second integral, when divided by μ, is ~~at least of~~ about the same as the order of the last integral, & $\mu \gg 1$. This is because $B_{ri}(R^-) = B_{ro}(R^+)$, for the normal component of $\vec{B}$ is continuous at a boundary.



Applying Gauss' law for the highly flattened donut-shaped Gaussian surface, as shown, we have

$$r\, B_{ro}(r) = R\, B_{ro}(R)$$

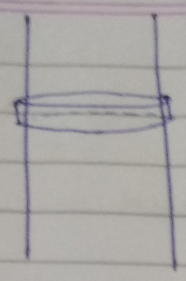
(to an approximation)

$$\Rightarrow B_{ro}(r) = \frac{R}{r} \cdot B_{ro}(R).$$

So our equation becomes :

$$\boxed{\frac{B_{ai}}{\mu} + R \ln\left(\frac{\lambda}{R}\right) \cdot \frac{\partial B_{ro}(R)}{\partial z} = 0}$$

Note that ~~this~~ the first approximation implies that B_{ai} is nearly constant in any ~~the~~ ~~inter~~ crossection which is perpendicular to the axis of the cylinder.



Applying Gauss' law for the shown cylinder gives us that

$$dz\left(\frac{\partial B_{ai}}{\partial z}\right)\pi R^2 + B_{ro}(R)\cdot dz\cdot 2\pi R = 0$$

$$\Rightarrow \boxed{\frac{R}{2}\left(\frac{\partial B_{ai}}{\partial z}\right) + B_{ro}(R) = 0.}$$

So using the previous equation, we put in the value of B_{ai} to get

$$\frac{R}{2}\left[\frac{\partial}{\partial z}\left(-\mu R \ln\left(\frac{\lambda}{R}\right)\cdot\frac{\partial B_{ro}(R)}{\partial z}\right)\right] + B_{ro}(R) = 0$$

$$\Rightarrow \quad +\frac{\mu R^2}{2}\ln\left(\frac{\lambda}{R}\right)\cdot\frac{\partial^2 B_{ro}(R)}{\partial z^2} = B_{ro}(R)$$

which is of the form

$$\frac{d^2y}{dx^2} = \omega^2 y, \quad (\omega > 0)$$

which has solutions of the form

$$y = Ae^{\omega x} + Be^{-\omega x}$$

Now since as $z \to \infty$, we expect $B_{ro}(R)$ to tend to 0, so $A = 0$.

Thus our expression for $B_{ro}(R)$ comes out to be

$$B_{ro}(R) = \left(B_{ro}(R)\Big|_{z=0^+}\right)\cdot \exp\left[-\sqrt{\frac{2}{\mu R^2 \ln\left(\frac{\lambda}{R}\right)}}\cdot z\right]$$

Since our characteristic scale was λ, we have

* Justified in the end

$$\sqrt{\frac{\mu R^2}{2} \ln\left(\frac{\lambda}{R}\right)} = \lambda$$

Now let $\left(\frac{\lambda}{R}\right)^2 = \alpha$; this will be used in determining the value of λ for large μ.

The relation above implies that

$$\frac{\mu}{4} \frac{\ln \alpha}{\alpha} = 1$$

$$\Rightarrow \quad \frac{\mu}{4} = \frac{\alpha}{\ln \alpha}$$

$$\Rightarrow \frac{\mu}{4} \ln \frac{\mu}{4} = \frac{\alpha}{\ln \alpha} \left(\ln \alpha - \ln \ln \alpha \right)$$

$$= \alpha \cdot \left[1 - \frac{\ln \ln \alpha}{\ln \alpha} \right]$$

Since $\mu \gg 1$ & $\ln \mu \gg 1$, $\frac{\ln \ln \alpha}{\ln \alpha} \sim 0$ [as $\alpha > \mu$]

So $\alpha = \frac{\mu}{4} \ln \frac{\mu}{4} \approx \frac{\mu}{4} \ln \mu$

(This is done since $\frac{\mu}{4} \ln \frac{\mu}{4} = \frac{\mu}{4} (\ln \mu - \ln 4)$

$$= \frac{\mu}{4} \ln \mu - \frac{\mu \ln 4}{4}.$$

and this is asymptotically equal to $\frac{\mu \ln \mu}{4}$).

$$\Rightarrow \frac{\lambda^2}{R^2} = \frac{\mu \ln \mu}{4}$$

$$\Rightarrow \boxed{\lambda \simeq \frac{R}{2} \sqrt{\mu \ln \mu}}$$

The first equation yields

$$B_{ai} = -\mu R \ln\left(\frac{\lambda}{R}\right) \frac{\partial B_{\phi 0}(R)}{\partial z}$$

$$= -\left(\frac{2\lambda^2}{R}\right) \cdot \frac{\partial B_{\phi 0}(R)}{\partial z}$$

$$= -\frac{2\lambda^2}{R} \cdot \left(B_{\phi 0}(R)\big|_{z=0}\right) \left(\frac{-1}{\lambda}\right) e^{-z/\lambda}$$

$$= \frac{2\lambda}{R} \cdot \left[B_{\phi 0}(R)\right]_{z=0} \cdot e^{-z/\lambda}$$

~~So the~~ Now the application of Ampere's circulation theorem along the axis gives

$$\mu \mu_0 i = \frac{2\lambda}{R} \cdot \left[B_{\phi 0}(R)\right]_{z=0} \left[2 \int_0^\infty e^{-z/\lambda}\, dz\right]$$

$$= \frac{4\lambda}{R} \cdot \left[B_{\phi 0}(R)\right]_{z=0} \int_0^\infty e^{-z/\lambda} \frac{dz}{\lambda} \cdot \lambda$$

$$= \frac{4\lambda^2}{R} \left[B_{\phi 0}(R)\right]_{z=0} \int_0^\infty e^{-x}\, dx$$

$$= \frac{4\lambda^2}{R} \left[B_{\phi 0}(R)\right]_{z=0}$$

$$= \cancel{4}R \cdot \frac{\mu \ln \mu}{\cancel{4}} \left[B_{\phi 0}(R)\right]_{z=0}$$

$$\Rightarrow \left[B_{\phi 0}(R)\right]_{z=0} = \frac{\mu \mu_0 i}{R \mu \ln \mu} = \frac{\mu_0 i}{R(\ln \mu)}$$

$$\Rightarrow B_{ai}(z=0) = \frac{2\lambda}{R} \cdot \frac{\mu_0 i}{R \ln \mu}$$

$$= \frac{2\lambda}{R^2} \frac{\mu_0 i}{\ln \mu}$$

$$= \frac{2}{R^2} \cdot \frac{R}{2} \cdot \sqrt{\mu \ln \mu} \cdot \frac{\mu_0 i}{\ln \mu}$$

$$= \frac{\mu_0 i}{R} \sqrt{\frac{\mu}{\ln \mu}}$$

$$\phi = \pi R^2 \cdot \frac{\mu_0 i}{R} \sqrt{\frac{\mu}{\ln \mu}} = Li$$

$$\Rightarrow \boxed{L = \mu_0 \pi R \cdot \sqrt{\frac{\mu}{\ln \mu}}}$$

which is the required self inductance.

Note: ① I have used R instead of the r in the problem statement. (This was just for the sake of convenience in remembering which ~~variable~~ name was allocated to a variable & which one for a constant).

② The justification for the usage of $\sqrt{\frac{\mu R^2}{2} \ln\left(\frac{\lambda}{R}\right)} = \lambda$ can be provided as follows:

If we don't use this value of λ in subsequent calculations, we obtain

$$L = \mu_0 \pi R \cdot \sqrt{\frac{\mu}{2 \ln\left(\frac{\lambda}{R}\right)}}$$

It is reasonable to expect $\sqrt{\frac{\mu R^2}{2}\ln\left(\frac{\lambda}{R}\right)} = \lambda \times k$ for some constant k, say, for example $k \leq 1000$ (This sort of a bound is necessary so that the dipole approximation after $R >$ some critical value works).

Then we have

$$\frac{\mu R^2}{2\lambda^2} \ln\left(\frac{\lambda}{R}\right) = k^2$$

$$\Rightarrow \frac{\mu}{4k^2} \frac{\ln \alpha}{\alpha} = 1$$

$$\Rightarrow \frac{\alpha}{\ln \alpha} = \frac{\mu}{4k^2}$$

$$\Rightarrow \alpha \approx \frac{\mu}{4k^2} \ln \mu.$$

$$\Rightarrow \frac{\lambda}{R} \approx \frac{\sqrt{\mu \ln \mu}}{2k}$$

$$\Rightarrow 2\ln\left(\frac{\lambda}{R}\right) \approx \frac{2\times 1}{2}\left(\ln \mu + \ln\ln\mu - \ln 4k^2\right)$$

where the first term dominates. So, we recover the result $L = \mu_0 \pi R \sqrt{\frac{\mu}{\ln \mu}}$.