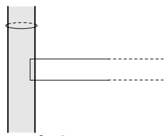
Physics Cup 2018 - Problem 3. June 6, 2018

Find the inductance of a circular loop of wire around an infinite ferromagnetic cylinder of radius r. The cylinder is made from a ferromagnetic material of relative permeability $\mu \gg 1$ (if needed, you can also assume that $\ln\mu \gg 1$); the radius of the loop is slightly larger than r so that it sits tightly around the cylinder.

The hints of 4th March – 8th April.

Keep in mind fact IX-30 from https://www.ioc.ee/~kalda/ipho/formulas.pdf. Also, it might be somewhat useful to read the solutions of http://www.ipho2012.ee/physicscup/problem-no-2/, see http://www.ipho2012.ee/physicscup/problem-no-2/solution/.

It appears to be convenient to write down differential equations for two unknown functions: (a) axial magnetic field inside the cylinder (the component parallel to the axis of the cylinder), and (b) radial magnetic field outside of the cylinder, near the surface of the cylinder. Although differential equations are involved, the mathematics is actually simple.



One of the two equations relating axial magnetic field inside the cylinder B_{ai} and radial magnetic field outside of the cylinder B_{ro} to each other can be obtained by writing down Ampère's circulation theorem for the loop shown in the figure. You still need to find a way for resolving the problem of diverging integrals along the radial infinite legs of this loop (you'll obtain a natural logarithm of something).

The both quantities (B_{ai} and B_{ro}) depend very slowly on the coordinate along the axis, let the characteristic scale be denoted by λ ($\lambda \gg r$). This characteristic scale is unknown at first, but you'll obtain an expression for it once you solve the two simple differential equations relating B_{ai} and B_{ro} to each other. Now, coming back to the above described logarithmic divergence: it is resolved using the scale λ similarly to how such a divergence is resolved when calculating the inductance of a piece of wire of length l and diameter d.

The other equations relating B_{ai} and B_{ro} to each other can be obtained by writing down Gauss theorem for a short cylinder tightly embracing the cylindrical rod. In this way, a set of two linear differential equations with constant multipliers can be obtained. These equations (and their solutions) are valid for any x (the coordinate along the axis) except for the neighbourhood of the wire loop at $x = 0$.

If everything goes well, you'll find that the magnetic field decays exponentially with $|x|$, i.e. is proportional to $\exp(-|x|/\lambda)$. Note that if the distance from the axis is approximately larger than λ, the magnetic field starts decaying with the distance from the axis R faster than $1/R$ because the cylindrically-symmetric solution is substituted with a dipole solution.

Final hints, 15th and 22nd April. Your solution needs to have the following components: (a) Understanding that $B_{ai}(x, R)$ and $B_{ro}(x, R))$ depend very slowly on the coordinate x along the axis, let the characteristic scale be denoted by λ ($\lambda \gg r$).

(b) Using Gauss theorem for a thin cylinder of height $h \ll \lambda$ and of radius R to deduce that $B_{ro}(x, R) \propto 1/R$.

(c) Writing down Ampère's circulation theorem for the loop shown in the figure (this will be referred to as Eq 1) and realizing that this will give us diverging integrals.

(d) Handling the divergence by noticing that the proportionality $B_{ro}(x, R) \propto 1/R$ holds only as long as $R \ll \lambda$ and that at larger distances, B_{ro} vanishes faster; hence, the diverging integral needs to be truncated at $R \approx \lambda$. (e) Writing down Gauss law for the magnetic field for a short cylinder which is coaxial with the ferromagnetic bar and has slightly larger radius than r (Eq 2). Eq 1 and 2 will give you a set of two linear differential equations with constant coefficients relating $B_{ai}(x, R = r)$ and $B_{ro}(x, R = r)$ to each other which you need to solve. (f) Writing down Ampère's circulation theorem for a loop extending along the axis of the ferromagnetic to infinity (and closing into a closed loop at infinity). This will relate $B_{ai}(x = 0, R = 0)$ to the current in the loop.

(g) Expressing the magnetic flux through the loop in terms of current and finding L as the coefficient of proportionality.

Correct solutions submitted:

name	**school**	**country**	**pr 3 solved**
Nozomi Sakura	Hiroo Gakuen High School	Japan	2 Apr 09:22
Thomas Bergamaschi	Colegio Etapa Valinhos-Brazil	Brazil	2 Apr 14:07
Navneel Singhal	ALLEN Kota	India	21 Apr 09:23
Dolteanu Stefan	Int. Comp. H-sch. Bucharest	Romania	29 Apr 09:20
Yunus Emre Parmaksiz	Bahcesehir H-sch. Sci & Tech.	Turkey	30 Apr 05:11
Mustafa Tugtekin	Bahcesehir H-sch. Sci & Tech.	Turkey	7 May 02:40