(I) Before describing the Brownian motion of the sphere, let us imagine that we put a force on the particle and investigate how would then it react against the force! Firstly, there is inertia, whose coefficient in this case is the mass, m' of the sphere. Secondly, if we kept a steady pull on the object, there would be a drag on it, proportional to its velocity. As we shall see, the presence of some irreversible loss in the system (like resistance) is essential in order that there be fluctuations. In this problem the drag force arises from the facts, that the moving sphere hits more photons on the front side and that its black body radiation is Doppler-shifted in a „standing" frame of reference. Let us find the coefficient γ in the equation $F_{drag} = -\gamma \cdot v$.

- Drag-force arising from Doppler-shift:
  - According to Planck's law the spectral radiance of a black body at a given temperature T:

$$B_\nu(\nu,T) = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{\frac{h\nu}{kT}}-1}$$

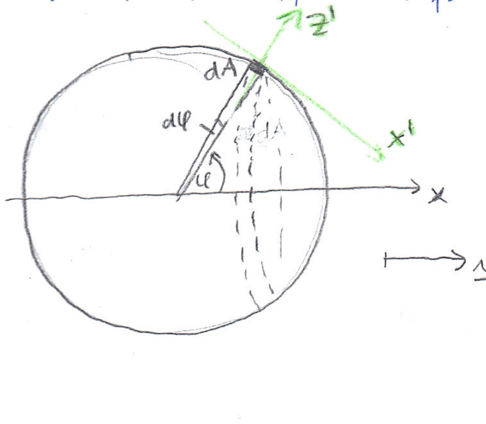
, which gives the power emitted per unit area of the body, per unit solid angle per unit frequency

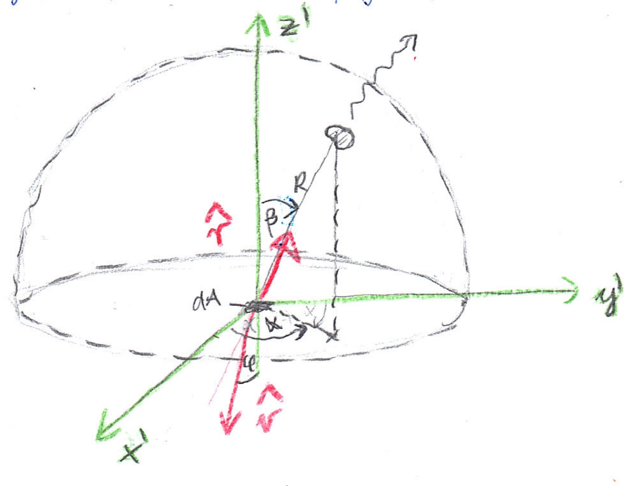
  - In the non-relativistic case the Doppler-shift is described by

$$f' = f_0\left(1+\frac{v}{c}\cos\Theta\right)$$

, where f_0 is the emitted frequency, f' is the measured frequency in the „standing" F.O.R., v is the speed of the sphere and Θ is the angle between the sphere's and the photon's velocity

  - Now let us consider a small surface element on the sphere (ΔA) characterised by angle φ. We will us the spherical polar coordinate system shown in the figures:





  - Firstly, we determine the angle Θ between unit vectors $\hat{r}$ and $\hat{v}$:

$$\hat{v} = (\sin\varphi;\ 0;\ \cos\varphi)$$

$$\hat{r} = (\cos\alpha\sin\beta;\ \sin\alpha\sin\beta;\ \cos\beta)$$

$$\cos\Theta = \hat{v}\cdot\hat{r} = \sin\varphi\cdot\cos\alpha\sin\beta + \cos\varphi\cdot\cos\beta$$

  - Now, the force acting on the sphere in the x' direction because of photons emitted in direction $\hat{r}$:

$$\left(\frac{dF_x}{dA}\right)\ dF_x^{+} = -\frac{1}{c}\cdot\left(B(\nu,T)\,d\nu\,d\Omega\cdot\cos\beta\, dA\right)\cdot\left(1+\frac{v}{c}\cos\Theta\right)\cdot\cos\Theta$$

Here the $\frac{1}{c}$ factor is due to the fact that the relation $p = \frac{E}{c}$ holds true for a photon, the $\cos\beta$ factor comes from Lambert's cosine law, the $\cos\theta$ factor is needed to calculate the x-component of the momentum.

- Knowing that $\cos\theta = \sin\varphi \cdot \cos\alpha \cdot \sin\beta + \cos\varphi \cdot \cos\beta$ and $d\Omega = \sin\beta \, d\alpha \, d\beta$ we gain:

$$\frac{dF_x^*}{dA} = -\frac{1}{c} \cdot B(\nu,T) \cdot \left(1 + \frac{v}{c}\left[\sin\varphi \cdot \cos\alpha \sin\beta + \cos\varphi \cos\beta\right]\right)\left[\sin\varphi\cos\alpha\sin\beta + \cos\varphi\cos\beta\right]\sin\beta\cos\beta \, d\nu \, d\beta \, d\alpha$$

- For simplicity we use the $I = \int_0^\infty B(\nu,T)\,d\nu = \frac{2\pi^4 k^4 T^4}{15 c^2 h^3} = \pi \cdot \sigma \cdot T^4$ notation. By integrating:

$$\frac{F_x^*}{dA} = -\frac{1}{c}\left[\int_{\nu=0}^{\infty} B(\nu,T)\,d\nu\right] \cdot \left[\int_{\alpha=0}^{2\pi}\int_{\beta=0}^{\pi/2}\left(1 + \frac{v}{c}\left[\sin\varphi\cos\alpha\sin\beta + \cos\varphi\cos\beta\right]\right)\left[\sin\varphi\cos\alpha\sin\beta + \cos\varphi\cos\beta\right]\sin\beta\cos\beta \, d\beta \, d\alpha\right] \Leftrightarrow$$

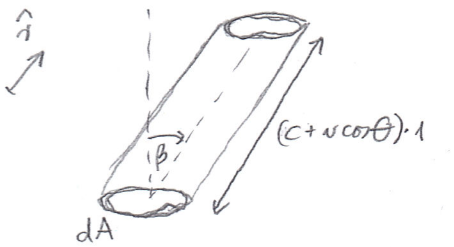
$$\Leftrightarrow -\frac{I}{c}\left[\frac{\pi}{8} \cdot \frac{v}{c}(\cos 2\varphi + 3) + \frac{2}{3}\pi\cos\varphi\right]$$

- Calculating the force for the entire sphere:

$$F_x = \int F_x^* \, dA = -\int_{\varphi=0}^{\pi} \frac{I}{c}\left[\frac{\pi}{8}\frac{v}{c}(\cos 2\varphi + 3) + \frac{2}{3}\pi\cos\varphi\right] \cdot 2R^2\sin\varphi \, d\varphi = -\frac{I}{c} \cdot \frac{4\pi}{3} R^2 \cdot \frac{v}{c}$$

$$\boxed{F_x = -\frac{4\pi^2}{3} \cdot \frac{R^2 \sigma T^4}{c^2} \cdot v} \Rightarrow$$

- <u>Drag-force arising from hitting more photon at the front side:</u>
  - Let us consider the photons hitting the surface area dA from/along the direction of $\hat{r}$ (see the previous derivation). Their density (number per unit time) and so the force they exert is proportional to the volume of the solid figure shown below:



- The relative speed of the surface element and the photon is (non-relativistically) $v_{rel} = c + v\cos\theta$
- The volume of the solid figure is $dA \cdot (c + v\cos\theta)\cos\beta$

$\rightarrow$ we have the same additional factors $\left(1 + \frac{v}{c}\right)$ and $\cos\beta$ as in the previous case

- So we can conclude, that the overall drag-force is $F_{drag} = -\frac{8\pi^2}{3} \cdot \frac{R^2 \sigma T^4}{c^2} \cdot v$, that is,

$$\boxed{\gamma = \frac{8\pi^2}{3} \cdot \frac{R^2 \sigma T^4}{c^2}}$$

(II) Now let's describe the Brownian motion! The formula for the motion under an external force F_{ext}, when we are pulling it in a normal manner, is

$$(1) \quad m\frac{d^2x}{dt^2} + \gamma \cdot \frac{dx}{dt} = F_{ext}$$

[In what follows we investigate a one-dimensional motion. If we work out the value of $\langle x^2 \rangle$, then $\langle r^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle = 3 \cdot \langle x^2 \rangle$ holds true in three dimensions.]

- To determine the mean square distance that the object goes during a time interval (much larger than the characteristic time between collisions) $\langle x^2 \rangle$ we use the same formula, but F_{ext} now is equal to the irregular forces from the collisions with photons. Let us multiply both sides of equation (1) by x and take time-average:

$$(2) \quad \left\langle m x \frac{d^2x}{dt^2} \right\rangle + \gamma \left\langle x \frac{dx}{dt} \right\rangle = \langle x F_x \rangle$$

- If the particle happens to have gone a certain distance x, then, since the irregular force is competely irregular and does not know where the particle started from, the next impulse can be in any direction relative to x. So $\langle x F_x \rangle = 0$.

The first term can be written as $\left\langle m x \cdot \frac{d^2x}{dt^2} \right\rangle = \left\langle m \cdot \frac{d\left[x \cdot \frac{dx}{dt}\right]}{dt} \right\rangle - \left\langle m \cdot \left(\frac{dx}{dt}\right)^2 \right\rangle$. Now x times the velocity has a mean that does not change with time, because when it gets to some position it has no remembrance of where it was before. This quantity, on the average, is zero. The average translational kinetic energy (corresponding to one degree of freedom) of the particle is known to be $\langle \frac{1}{2} m v_x^2 \rangle = \frac{1}{2} kT$ based on the equipartition theorem.

The term $\langle x \frac{dx}{dt} \rangle$ can also be written as $\langle \frac{1}{2} \frac{d(x^2)}{dt} \rangle$.

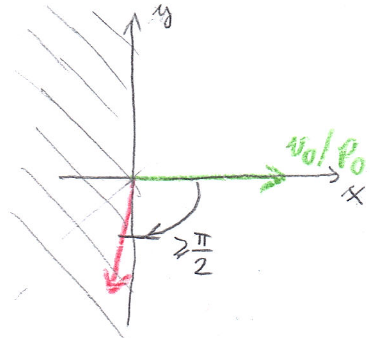
- Putting together what we have so far:

$$(3) \quad -\langle m v_x^2 \rangle + \gamma/2 \cdot \frac{d}{dt}\langle x^2 \rangle = 0$$

$$\frac{d}{dt}\langle x^2 \rangle = 2\frac{kT}{\gamma} \Rightarrow \langle x^2 \rangle = \frac{2kT}{\gamma} \cdot t$$

- Therefore the sphere has a mean square distance $\boxed{\langle r^2 \rangle = 3\langle x^2 \rangle = \frac{6kT}{\gamma} \cdot t}$

(III) Our last task is to find the typical time $t_{\pi/2}$ required for the particle's velocity to turn by an angle $\pi/2$. The typical ~~velocity~~ speed of the particle is $\sqrt{\langle v^2 \rangle} = v_0 = \sqrt{3\frac{kT}{m}}$ (this might differ from the exact value ~~with a numerical~~ of $\langle v \rangle$ with a numerical factor in the order of one, however, we seek only an estimation and as we will see, working with v_0 is sufficient enough). Let the direction of x-axis be the same of the particle's velocity at t=0. Now the condition for the velocity to turn by an angle $\pi/2$ is equivalent to that v_x component (or alternatively p_x, the x-component of the momentum) must reach zero (or change signe).



- Let $\bar{p}$ be the most probable momentum of one photon. According to Wien's law, this is $\left(\frac{x}{h} kT\right)\frac{h}{c}$ where $x \simeq 2.82$ – we will leave the numerical factor so $\bar{p} \simeq \frac{kT}{h} \cdot \frac{h}{c} = \frac{kT}{c}$

(We could also use average momentum of one photon, however, it is hard to determine because of the integral $\int \frac{\nu^2}{e^{\frac{h\nu}{kT}}-1} d\nu$ needed to calculate the overall number of photons)

- For a Brownian-particle, the ~~aver~~ typical momentum is much larger than the change in the momentum during one collition (the short atomic scale T_S describing the collitsions and the Brownian timescale $T_B \approx \frac{m}{\gamma}$ for the relaxation of the particle velocity, moreover, the relaxation time T_r for the Brownian particle statisfy $T_S \ll T_B \ll T_r$). As long as $p_x \gg \bar{p}$, the ~~co~~ differential equation $\frac{dp_x}{dt} = -\gamma \frac{p(t)}{m}$ describes the system quite well (on average), suggesting that

$$(4) \quad p_x(t) = p_0 \cdot e^{-t/T_B} \text{ where } p_0 \approx m v_0 \approx \sqrt{3 k T m} \text{ and } T_B = \frac{m}{\gamma}$$

- According to this, the velocity and the momentum of the particle is predicted to decay to zero at long times. However, this cannot be true, as we know that $\langle v_x^2 \rangle = \frac{kT}{m}$ in equilibrium. When p_x reaches the order of $\bar{p}$, then the random nature of the collisions cannot be handled by averaging: only one photon might changes p_x to zero or cause a change in p_x's sign.
- So we can estimate the typical time we are looking for as the time needed to reach $\bar{p}$ according to the formula $p_x(t) = p_0 \cdot e^{-t/T_B}$

$$\Rightarrow t_{\pi/2} \approx T_B \cdot \ln\left(\frac{p_0}{\bar{p}}\right) = \frac{m}{\gamma} \cdot \ln\left(\frac{\sqrt{3kTm}}{\frac{kT}{c}}\right) = \boxed{\frac{1}{2} \frac{m}{\gamma} \cdot \ln\left(\frac{3mc^2}{kT}\right)}$$

(IV) Finally, we are ready to determine the mean-free path:

$$s = \sqrt{\frac{6kT}{\gamma} \cdot t_{\pi/2}} = \sqrt{\frac{6kT}{\gamma} \cdot \frac{1}{2} \frac{m}{\gamma} \cdot \ln\left(\frac{3mc^2}{kT}\right)} = \sqrt{3mkT \cdot \ln\left(\frac{3mc^2}{kT}\right)} \cdot \frac{3}{8\pi^2} \cdot \frac{c^2}{R^2 \sigma T^4}$$

$$s = \frac{3\sqrt{3}}{8\pi^2} \cdot \frac{c^2}{R^2 \sigma T^4} \cdot \sqrt{(mkT) \cdot \ln\left(3 \cdot \frac{mc^2}{kT}\right)}$$

Please note that the numerical factors $\frac{3\sqrt{3}}{8\pi^2}$ and 3 might differ from what a more complicated and more precise mathematical analysis would give, however, the derivated formula for s above is only an approximation.