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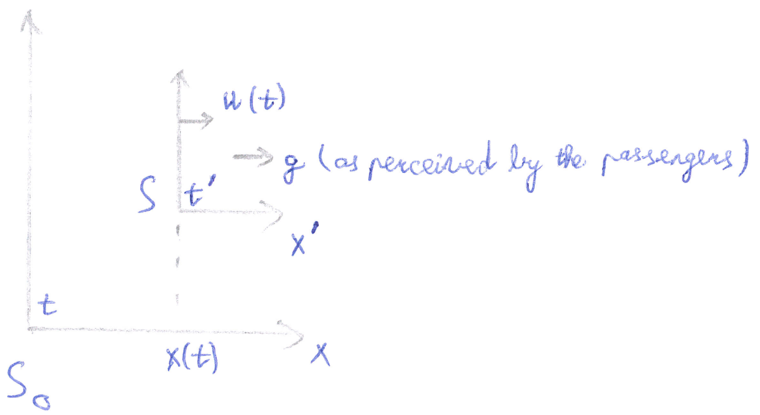
τ - ?

$v,\ g,\ c$

We are going to consider two frames of reference:

S_0 – the ~~one~~ inertial frame moving at a ~~constant~~ speed of the spaceship the moment it launches the missiles; $t = 0$ corresponds to the moment the missiles are launched.

S – accelerating frame in which the spaceship is always at rest.



For v comparable to c, the only way to measure acceleration in the spaceship is by measuring fictitious force $m_0 g$ acting on a body (stationary in S) of rest mass m_0.

On the other hand, this force is equal to the rate of change of momentum, as measured in S_0:

$$\frac{dp}{dt} = \dot{p} = m_0 g, \quad \text{where } p = \frac{m_0 u}{\sqrt{1 - \frac{u^2}{c^2}}} \quad (1)$$

Therefore, $p(t) = \int_0^t m_0 g \, dt_1 = m_0 g t$. Substitute (1):

$$\frac{m_0 u}{\sqrt{1 - \frac{u^2}{c^2}}} = m_0 g t \quad \Rightarrow \quad u(t) = \frac{g c t}{\sqrt{c^2 + g^2 t^2}} \quad (2)$$

~~Then~~ $\cancel{x(t) = \int_0^t g c t_1}$

Then $x(t) = \int_0^t u(t_1)\, dt_1 = \int_0^t \frac{g c t_1 \, dt_1}{\sqrt{c^2 + g^2 t_1^2}} = \frac{c}{2g} \int_{t_1=0}^{t} \frac{d(c^2 + g^2 t_1^2)}{\sqrt{c^2 + g^2 t_1^2}} = \frac{c}{g} \left[\sqrt{c^2 + g^2 t_1^2} \right]_{t_1=0}^{t} =$

$$= \frac{c^2}{g}\left[\sqrt{1+\frac{g^2t^2}{c^2}}-1\right].$$

If the 1st ~~rocket~~ missile was caught at time t_I (as measured in S_0), then

$$x(t_I) = v\,t_I;$$

$$\frac{c^2}{g}\left[\sqrt{1+\frac{g^2t_I^2}{c^2}}-1\right] = v\,t_I \Rightarrow t_I = \frac{2v}{g\left(1-\frac{v^2}{c^2}\right)} \quad (3)$$

Analogically, the time at which the 2nd missile was caught is found from

$$x(t_{II}) = 2v\,t_{II} \Rightarrow t_{II} = \frac{4v}{g\left(1-\frac{4v^2}{c^2}\right)} \quad (4)$$

Lorentz time transformation for non-accelerating frames:

$$t' = \frac{t-\frac{xv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}.$$ We can apply it for a small time interval dt at time t:

$$dt' = \frac{dt-\frac{dx\,\cancel{u}u(t)}{c^2}}{\sqrt{1-\frac{u(t)^2}{c^2}}} = \frac{dt-\frac{u(t)^2\,dt}{c^2}}{\sqrt{1-\frac{u(t)^2}{c^2}}} = \sqrt{1-\frac{u(t)^2}{c^2}}\,dt.$$

Thus, the total proper time interval

$$\tau = \int_{t=t_I}^{t_{II}} dt' = \int_{t_I}^{t_{II}} \sqrt{1-\frac{u(t)^2}{c^2}}\,dt.$$

Substitute (2):

$$\tau = \int_{t_I}^{t_{II}} \sqrt{1-\frac{1}{c^2}\frac{g^2c^2t^2}{(c^2+g^2t^2)}}\,dt = \int_{t_I}^{t_{II}} \frac{dt}{\sqrt{1+\left(\frac{gt}{c}\right)^2}}$$

~~Let φ = arccos~~ Define $\varphi \equiv \operatorname{arcsinh}\left(\frac{gt}{c}\right) \Rightarrow \sinh\varphi = \frac{gt}{c} \Rightarrow \cosh\varphi\, d\varphi = g\frac{dt}{c}$.

Then $$\tau = \int_{t=t_I}^{t_{II}} \frac{c}{g}\frac{\cosh\varphi\, d\varphi}{\sqrt{1-\sinh^2\varphi}} = \int_{t=t_I}^{t_{II}} \frac{c}{g}\,d\varphi = \frac{c}{g}(\varphi)\Big|_{t=t_I}^{t_{II}} = \frac{c}{g}\left(\operatorname{arcsinh}\left(\frac{gt}{c}\right)\right)\Big|_{t_I}^{t_{II}}$$

Substitute (3) and (4): $$\boxed{\tau = \frac{c}{g}\left[\operatorname{arcsinh}\left(\frac{4\frac{v}{c}}{1-\left(\frac{2v}{c}\right)^2}\right) - \operatorname{arcsinh}\left(\frac{2\frac{v}{c}}{1-\left(\frac{v}{c}\right)^2}\right)\right]}$$