Solution for Problem 2

Satoshi Yoshida

If B is moving with speed v_B in the A's frame and A is moving with speed v_A in the same direction as B in the frame O, then the speed of A in the frame O is $v'_B = \frac{v_A+v_B}{1+\frac{v_A v_B}{c^2}}$. Let $v_A = c\tanh u_A$, $v_B = c\tanh u_B$ and $v'_B = c\tanh u'_B$. Then, $u'_B = u_A + u_B$ because of the addition theorem of hyperbolic tangent. I call u_A, u_B and u'_B rapidity. For example, u_A is the rapidity of A in the frame O.

Let τ be the proper time of the spaceship and two missiles are launched at $\tau = 0$. Let O be the inertial frame that is moving at the same speed as the spaceship at $\tau = 0$. Let t be the time in the frame O and $t = 0$ at $\tau = 0$. Let v_S be the speed of the spaceship and u_S be the rapidity of the space ship in the frame O. The spaceship catches the first and the second missile at $\tau = \tau_1, \tau_2$, respectively. $t = t_1, t_2$ at $\tau = \tau_1, \tau_2$, respectively.

If the speed v is small, the rapidity is $u \cong \frac{v}{c}$ because $v = c\tanh u \cong cu$. Therefore,

$$\frac{\mathrm{d}u_S}{\mathrm{d}\tau} = \frac{g}{c}$$

At $\tau = 0$, the speed of the spaceship in the frame O is 0. Then,

$$u_S = \frac{g}{c}\tau$$

$$v_S = c\tanh u_S = c\tanh\frac{g}{c}\tau$$

Because the first missile is caught at $t = t_1$,

$$\int_0^{t_1} v_S \mathrm{d}t = \int_0^{t_1} v \mathrm{d}t$$

Substituting $\mathrm{d}t = \frac{\mathrm{d}\tau}{\sqrt{1-\left(\frac{v_S}{c}\right)^2}} = \cosh\frac{g}{c}\tau\,\mathrm{d}t$ and $v_S = c\tanh\frac{g}{c}\tau$,

$$\int_0^{\tau_1} c\sinh\frac{g}{c}\tau\,\mathrm{d}\tau = \int_0^{\tau_1} v\cosh\frac{g}{c}\tau\,\mathrm{d}\tau$$

$$\frac{c^2}{g}\left(\cosh\frac{g}{c}\tau_1 - 1\right) = \frac{cv}{g}\sinh\frac{g}{c}\tau_1$$

Let $\xi = \exp\frac{g}{c}\tau_1$, then,

$$c\left(\frac{\xi+\xi^{-1}}{2} - 1\right) = v\frac{\xi-\xi^{-1}}{2}$$

$$(c-v)\xi^2-2c\xi+c+v=0$$

$$\xi=\frac{c\pm v}{c-v}$$

Because $\tau_1>0$, $\xi>1$. Then,

$$\xi=\frac{c+v}{c-v}$$

Therefore,

$$\tau_1=\frac{c}{g}\log\frac{c+v}{c-v}$$

Similarly,

$$\tau_2=\frac{c}{g}\log\frac{c+2v}{c-2v}$$

In conclusion, the proper time interval in the spaceship between catching the first missile and catching the second missile is

$$\tau_2-\tau_1=\frac{c}{g}\left(\log\frac{c+2v}{c-2v}-\log\frac{c+v}{c-v}\right)$$