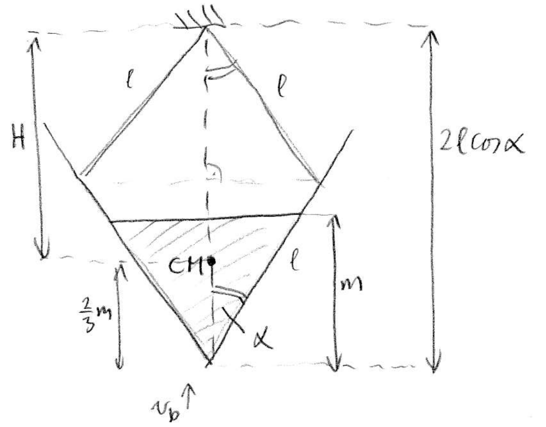
- The equilibrium-point of the system can be found from the potential energy of the system, which is minimal (for a given amount of water).

- figure 1:



- the total volume of water: $V = Lm^2 \operatorname{tg}\alpha \Rightarrow m = \sqrt{\frac{V}{L\operatorname{tg}\alpha}} = \sqrt{\frac{S}{\operatorname{tg}\alpha}}$

  where $\frac{V}{L} \equiv S$ is constant

- the distance of the centre of mass from the ceiling:

  $$H = 2l\cos\alpha - \frac{2}{3}m = 2l\cos\alpha - \frac{2}{3}\sqrt{\frac{S}{\operatorname{tg}\alpha}}$$

- the gravitational potential energy is minimal when $\left.\frac{dH}{d\alpha}\right|_{\alpha=\alpha_0} = 0$

  $$(1) \quad -2l\sin\alpha_0 + \frac{1}{3}\sqrt{S\operatorname{tg}\alpha_0}\left(1+\frac{1}{\operatorname{tg}^2\alpha_0}\right) = 0$$

- the water almost filles up the ~~entire~~ vesel, so:

  $$(2) \quad \frac{V}{L} = S = l^2 \sin\alpha_0 \cos\alpha_0$$

- (1) and (2): $-2l\sin\alpha_0 + \frac{1}{3}\sqrt{l^2\sin\alpha_0\cos\alpha_0 \cdot \frac{\sin\alpha_0}{\cos\alpha_0}}\left(1+\frac{1}{\operatorname{tg}^2\alpha_0}\right) = 0$

  $$-2l\sin\alpha_0 + \frac{1}{3}l\sin\alpha_0\left(1+\frac{1}{\operatorname{tg}^2\alpha_0}\right) = 0 \qquad ! \sin\alpha_0 \neq 0$$

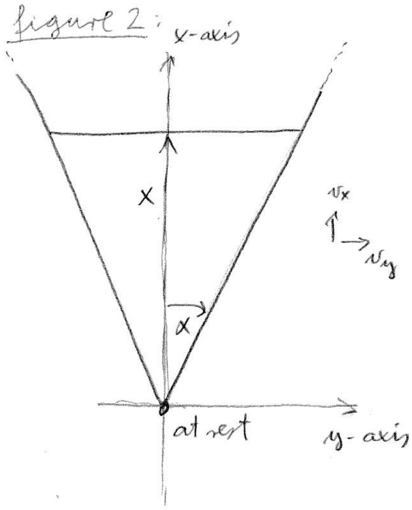
  $$-2 + \frac{1}{3}\left(1+\frac{1}{\operatorname{tg}^2\alpha_0}\right) = 0$$

  $$\operatorname{tg}\alpha_0 = \frac{1}{\sqrt{5}} \qquad ! \; 0 < \alpha_0 < \frac{\pi}{2}$$

- $\boxed{\text{The angle between the plates: } 2\alpha_0 = 2\cdot\operatorname{arctg}\left(\frac{1}{\sqrt{5}}\right) \approx 0{,}841\,\text{rad} \approx 48{,}19^\circ}$

- Now let us relate the speed of the water particles to the angular velocity of the vessel ($\dot{\alpha}$). Let the bottom point of the vessel be at rest. According to the first hint if we draw a set of equispaced horisontal lines onto the water, they preserve these properties. This indicates that the vertical velocity of water particles are the same in a horizontal plane. The volume of water under shuch a line is constant.

- figure 2:



- $\frac{V(x)}{L} = x^2 \cdot \operatorname{tg}\alpha = S(x)$ is constant

- derivating with respect to time (and taking values around the e. p.)

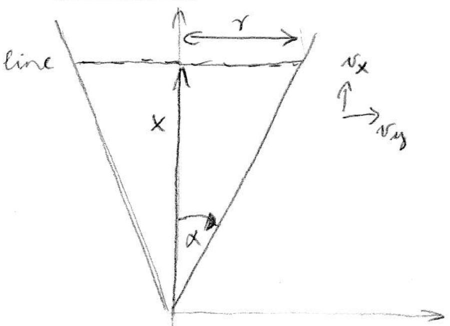
  $$2x \cdot \frac{dx}{dt}\operatorname{tg}\alpha_0 + x^2 \cdot \frac{1}{\cos^2\alpha_0}\cdot\frac{d\alpha}{dt} = 0$$

  $$v_x(x) = \frac{dx}{dt} = -\frac{1}{2}\frac{1}{\operatorname{tg}\alpha_0\cos^2\alpha_0}\cdot x\cdot\dot{\alpha} = -\frac{3}{\sqrt{5}}x\cdot\dot{\alpha}$$

  $$\left(\operatorname{tg}\alpha_0 = \frac{1}{\sqrt{5}} \Rightarrow \sin\alpha_0 = \frac{1}{\sqrt{6}}\;;\; \cos\alpha_0 = \sqrt{\frac{5}{6}}\right)$$

• As the oscillation is ~~symmetric to)~~ mirror-symmetric, the horizontal velocity of water particles at $y=0$ is zero. Water particles near the plates follow its motion horizontally, so horizontal velocity at these point also can be determined:

• figure 3:



• $\frac{V}{L} = x^2\, tg\alpha = \frac{r^2}{tg\alpha} = S(x) = \text{const.} \quad (r = x \cdot tg\alpha)$

• taking derivatives of both sides with respect to time:

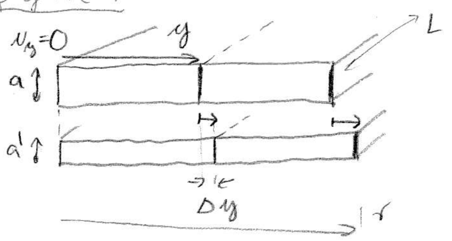
$$\frac{2r\frac{dr}{dt}\cdot tg\alpha_0 - r^2\cdot\frac{1}{\cos^2\alpha_0}\frac{d\alpha}{dt}}{tg^2\alpha} = 0 \qquad (tg\alpha \neq 0)$$

$$\frac{dr}{dt} = v_y(x, x\,tg\alpha) = +\frac{1}{2}\,\frac{1}{tg\alpha_0\,\cos^2\alpha_0}\cdot r\dot{\alpha} = +\frac{3}{\sqrt{5}}(x\,tg\alpha_0)\cdot\dot{\alpha}$$

(holds true for the $y \geq 0$ region)

• Considering the water between two ~~dx~~ adjacent horizontal line one can easily derive that the horizontal velocity must increase linearly with the distance from x-axis (the two boundary plates remain parallel and there is no water flux throug the plates)

• figure 4:



• $a y L = \text{const} \Rightarrow \frac{da}{dt}y + a\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{1}{a}\frac{da}{dt}\,y \propto y$

• $v_y(x,0) = 0$; $v_y(x, y = x\,tg\alpha) = +\frac{3}{\sqrt{5}}(x\,tg\alpha_0)\dot{\alpha} \Rightarrow v_y(x,y) = +\frac{3}{\sqrt{5}}y\dot{\alpha}$ (holds true for the entire region)

• The bottom of the vessel moves vertically with respect to the ceiling, its speed is:

$$v_0 = -\frac{d}{dt}(2l\cos\alpha) = +2l\sin\alpha_0\cdot\dot{\alpha} = \frac{2}{\sqrt{6}}l\dot{\alpha}$$

• So the velocity of a water particle with respect to the ceiling:

$$v_x(x,y) = v_0 + v_x'(x) = \frac{2}{\sqrt{6}}l\dot{\alpha} - \frac{3}{\sqrt{5}}x\dot{\alpha} \quad ; \quad v_y(x,y) = +\frac{3}{\sqrt{5}}y\dot{\alpha}$$

• The total kinetic energy of the water:

$$dK = \frac{1}{2}(L\,dx\,dy\,\varrho)\left(v_x^2(x,y) + v_y^2(x,y)\right) = \frac{1}{2}(L\,dx\,dy\,\varrho)\left[\left(\frac{2}{\sqrt{6}}l - \frac{3}{\sqrt{5}}x\right)^2\dot{\alpha}^2 + \left(\frac{3}{\sqrt{5}}y\right)^2\dot{\alpha}^2\right]$$

$$K = 2\cdot\int_0^{l\cos\alpha_0}dx\int_0^{x\cdot tg\alpha_0}dy\left\{\frac{1}{2}L\varrho\dot{\alpha}^2\left[\left(\frac{2}{\sqrt{6}}l - \frac{3}{\sqrt{5}}x\right)^2 + \left(\frac{3}{\sqrt{5}}y\right)^2\right]\right\} = L\varrho\dot{\alpha}^2\int_0^{\sqrt{\frac{5}{6}}l}dx\int_0^{\frac{1}{\sqrt{5}}x}dy\left\{\left(\frac{2}{\sqrt{6}}l - \frac{3}{\sqrt{5}}x\right)^2 + \left(\frac{3}{\sqrt{5}}y\right)^2\right\} =$$

$$= L\varrho\dot{\alpha}^2\int_0^{\sqrt{\frac{5}{6}}l}\left\{\left(\frac{2}{\sqrt{6}}l - \frac{3}{\sqrt{5}}x\right)^2\cdot\frac{x}{\sqrt{5}} + \frac{3}{25\sqrt{5}}x^3\right\}dx = L\varrho\dot{\alpha}^2 l^4\cdot\frac{\sqrt{5}}{90}$$

- Let us introduce $\xi := \alpha - \alpha_0$
- The total kinetic energy of the water: $K = L\varrho l^4 \cdot \frac{\sqrt{5}}{90} \cdot \dot{\xi}^2$
- Now let us express the gravitational potential energy around the equilibrium point as a function of ξ ($\xi \ll 1$) up to second order (we choose the potential energy zero at the e. point):
  - distance of the centre of mass from the ceiling: $H = 2l\cos\alpha - \frac{2}{3}\sqrt{\frac{S}{\operatorname{tg}\alpha}}$ (S is constant)
  - $\left.\frac{dH}{d\alpha}\right|_{\alpha=\alpha_0} = 0, \quad \left.\frac{d^2H}{d\alpha^2}\right|_{\alpha=\alpha_0} = -\frac{4}{5}\sqrt{30}\cdot l$

$$\Rightarrow \Pi(\xi) = -H(\xi)Mg - (-H(\xi=0)Mg) = -\left(H(0) + \left.\frac{dH}{d\alpha}\right|_{\alpha=\alpha_0}\cdot\xi + \frac{1}{2}\left.\frac{d^2H}{d\alpha^2}\right|_{\alpha=\alpha_0}\xi^2\right)Mg + H(0)Mg = +\frac{2}{5}\sqrt{30}\, l\, \xi^2 \cdot Mg$$

where $M = (SL)\varrho = l^2 \sin\alpha_0 \cos\alpha_0 L\varrho = \frac{\sqrt{5}}{6} l^2 L \varrho$

- The total energy of the system (kinetic + potential):

$$E = K + \Pi = \frac{\sqrt{5}}{90}\cdot L\varrho l^4 \dot{\xi}^2 + \frac{\sqrt{6}}{3} L\varrho l^3 g \cdot \xi^2$$

- If there is no friction in the water and between the water and the plates, E is conserved. Taking time derivatives of both sides:

$$0 = \frac{\sqrt{5}}{90} L\varrho l^4 \cdot 2\cancel{\dot{\xi}}\ddot{\xi} + \frac{\sqrt{6}}{3} L\varrho l^3 g \cdot 2\xi\cancel{\dot{\xi}}$$

$$0 = \frac{2\sqrt{5}}{90} l \cdot \ddot{\xi} + \frac{2\sqrt{6}}{3} g \cdot \xi$$

$$\ddot{\xi} = -\frac{g}{l}\cdot 6\sqrt{30}\cdot\xi$$

- $\boxed{\text{The angular frequency of the oscillation: } \omega = \sqrt{\frac{g}{l}}\cdot\sqrt{6\sqrt{30}} \approx 5{,}733\sqrt{\frac{g}{l}}}$