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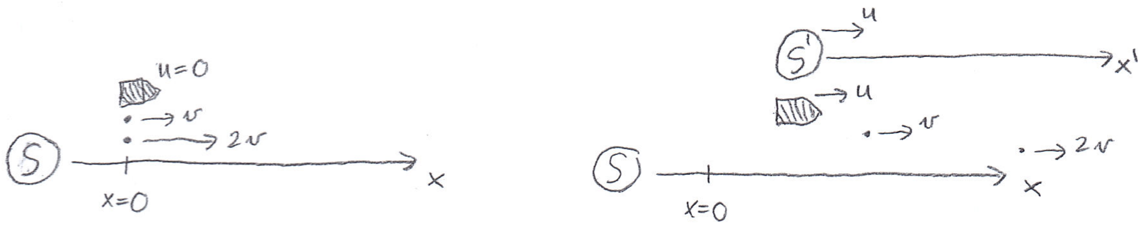
- Let S be an external inertial frame of reference such that the relative velocity of the spaceship and S is zero at the moment of the missiles' launch. Then the missiles have velocities v and $2v$ in S, too. Furthermore, let us choose the coordinate time and the position of the spaceship in S so that at the moment of the missiles's launch $t=0$ and $x_s(t=0)=0$. The velocity and the acceleration of the spaceship (as seen from S) are $u(t)=\frac{dx_s(t)}{dt}$ and $a(t)=\frac{d}{dt}u(t)=\frac{d^2}{dt^2}x_s(t)$.



- Let S' be an inertial frame of reference in which the spaceship is momentarily at rest. The proper acceleration of the spaceship is $a'=g$ (measured in S'). According to the formula of acceleration transformation,[1] the spaceship's acceleration in S is:

$$(1)\quad a = a'\cdot\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}} = g\cdot\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}} = \frac{du}{dt}$$

- Solving the separable differential equation:

$$\int_0^{u(t)} \frac{du}{\sqrt{1-\frac{u^2}{c^2}}^{\,3}} = \int_0^t g\,d\tilde{t} \iff \left[\frac{u}{\sqrt{1-\frac{u^2}{c^2}}}\right]_0^{u(t)} = \left[g\tilde{t}\right]_0^t \iff \frac{u(t)}{\sqrt{1-\frac{u^2(t)}{c^2}}} = g\cdot t$$

Thus:

$$(2)\quad u(t) = \frac{g\cdot t}{\sqrt{1+\left(\frac{g}{c}\right)^2 t^2}},$$

and

$$(3)\quad x_s(t) = \int_0^t u(\tilde{t})\,d\tilde{t} = \left[\frac{c^2}{g}\sqrt{1+\left(\frac{g}{c}\right)^2\tilde{t}^2}\right]_0^t = \frac{c^2}{g}\left(\sqrt{1+\left(\frac{g}{c}\right)^2 t^2}-1\right)$$

- The position of the missiles as a function of coordinate time:

$$(4)\quad x_{m1}(t) = \int_0^t v\,d\tilde{t} = v\cdot t \quad ; \quad x_{m2}(t) = \int_0^t 2v\,d\tilde{t} = 2v\cdot t$$

- Let the spaceship catch the first missile at $t=T_1$. We have

$$(5)\quad x_s(T_1) = x_{m1}(T_1) \iff \frac{c^2}{g}\left(\sqrt{1+\left(\frac{g}{c}\right)^2 T_1^2}-1\right) = v\cdot T_1 \iff \sqrt{1+\left(\frac{g}{c}\right)^2 T_1^2} = \frac{gv}{c^2}\cdot T_1 + 1$$

Equation (5) leads to a quadratic equation with roots $T_1=0$ and

$$(6)\quad T_1 = \frac{2\frac{v}{g}}{1-\frac{v^2}{c^2}}$$

- Similarly, for the other missile (with speed $2v$) we get:

$$(7)\quad T_2 = \frac{4\frac{v}{g}}{1-4\cdot\frac{v^2}{c^2}}$$

- So far we found the time interval between catching the missiles in the frame of reference S: $\Delta t = T_2 - T_1$. To find the proper time interval (measured in the spaceship's frame of reference), let us consider an infinitesimal time interval dt (in S). During this interval the spaceship's velocity is approximately constant, so according to the formula of time dilation the small time interval measured on the spaceship is

$$(8)\quad dT = dt\cdot\sqrt{1-\frac{u^2}{c^2}}$$

- By integratin upon time

$$(9)\quad \Delta T = \int_{T_1}^{T_2}\sqrt{1-\frac{u^2(t)}{c^2}}\,dt \overset{(2)}{=} \int_{T_1}^{T_2}\frac{1}{\sqrt{1+\left(\frac{g}{c}\right)^2 t^2}}\,dt = \left[\frac{c}{g}\cdot\ln\left(\sqrt{\left(\frac{g}{c}\right)^2 t^2+1}+\left(\frac{g}{c}\right)t\right)\right]_{T_1}^{T_2} \ominus$$

$$\ominus\ \frac{c}{g}\cdot\ln\frac{\sqrt{1+\left(\frac{g}{c}\right)^2 T_2^2}+\left(\frac{g}{c}\right)T_2}{\sqrt{1+\left(\frac{g}{c}\right)^2 T_1^2}+\left(\frac{g}{c}\right)T_1} \overset{(6,7)}{=} \frac{c}{g}\cdot\ln\frac{\dfrac{\left(\frac{2v}{c}\right)^2+2\cdot\left(\frac{2v}{c}\right)+1}{1-\left(\frac{2v}{c}\right)^2}}{\dfrac{\left(\frac{v}{c}\right)^2+2\cdot\left(\frac{v}{c}\right)+1}{1-\left(\frac{v}{c}\right)^2}} = \frac{c}{g}\cdot\ln\frac{\left(\frac{2v}{c}+1\right)^2\left(1+\frac{v}{c}\right)\left(1-\frac{v}{c}\right)}{\left(\frac{v}{c}+1\right)^2\left(1+\frac{2v}{c}\right)\left(1-\frac{2v}{c}\right)} \ominus$$

$$\ominus\ \frac{c}{g}\cdot\ln\frac{\left(1+\frac{2v}{c}\right)\left(1-\frac{v}{c}\right)}{\left(1+\frac{v}{c}\right)\left(1-\frac{2v}{c}\right)} = \frac{c}{g}\cdot\ln\left(\frac{1+\left(\frac{v}{c}\right)-2\left(\frac{v}{c}\right)^2}{1-\left(\frac{v}{c}\right)-2\left(\frac{v}{c}\right)^2}\right)$$

- Conclusion:

The proper time interval in the spaceship between catching the first missile and catching the second missile is:

$$\Delta T = \frac{c}{g}\cdot\ln\left(\frac{1+\left(\frac{v}{c}\right)-2\left(\frac{v}{c}\right)^2}{1-\left(\frac{v}{c}\right)-2\left(\frac{v}{c}\right)^2}\right)$$

- Sanity check of the result:
  - $\Delta T(v)$ can be evaluated for any valid v $(0 \le v < 0.5c)$ and is a monotonically increasing function of v on the interval in question
  - $\lim_{2v\to c}\Delta T(v) = +\infty$; $\Delta T(v=0)=0$
  - If $v \ll c \Leftrightarrow \frac{v}{c} \ll 1$, then using a first-order Taylor series approximation:

$$\Delta T(v) \approx \Delta T(0) + \left.\frac{d\,\Delta T}{dv}\right|_{v=0}\cdot v = \frac{2v}{g}$$

This agrees with the result one gets using classical mechanics.

- References: [1] https://en.wikipedia.org/wiki/Acceleration_(special_relativity)#cite_note-lorentz1-4