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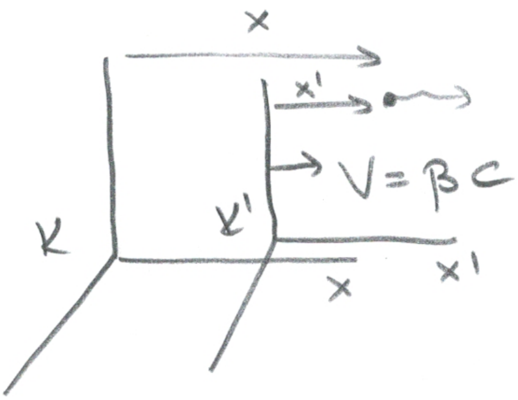
Let us investigate the relativistic motion in 1-dimension.



$(\tau = ct,\ \tau' = ct')$

Lorentz transformation is given:

$$\begin{cases} x' = \gamma(x - \beta\tau) \\ \tau' = \gamma(\tau - \beta x) \end{cases};$$

where $\gamma = \dfrac{1}{\sqrt{1-\beta^2}} = \dfrac{1}{\sqrt{1 - V^2/c^2}}$ is the Lorentz factor.

$$v' = c \cdot \frac{dx'}{d\tau'} = c \cdot \frac{\gamma(dx - \beta d\tau)}{\gamma(d\tau - \beta dx)}$$

$$= c \cdot \frac{dx/d\tau - \beta}{1 - \beta\, dx/d\tau} = c \cdot \frac{v/c - \beta}{1 - \beta v/c}$$

$$\begin{cases} v' \equiv \beta_{v'} \cdot c \\ v \equiv \beta_v \cdot c \end{cases} \Longrightarrow \boxed{\beta_{v'} = \frac{\beta_v - \beta}{1 - \beta\beta_v}} \quad (1)$$

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$$\begin{cases} a' = \dfrac{d^2x'}{dt'^2} = c^2 \dfrac{d}{d\tau'}\left(\dfrac{dx'}{d\tau'}\right) = c^2 \dfrac{d\beta_\vartheta'}{d\tau'} \Rightarrow \\ a = c^2 \dfrac{d\beta_\vartheta}{d\tau} \end{cases}$$

$$\frac{d\beta_\vartheta'}{d\tau'} = \frac{1}{\gamma(d\tau - \beta dx)} \cdot \frac{d\beta_\vartheta(1-\beta\beta_\vartheta) + (\beta_\vartheta - \beta)\beta d\beta_\vartheta}{(1-\beta\beta_\vartheta)^2}$$

$$a' = \frac{a(1 - \cancel{\beta\beta_\vartheta} + \cancel{\beta\beta_\vartheta} - \beta^2)}{\gamma(1-\beta\beta_\vartheta)^3} = \frac{a}{\gamma^3(1-\beta\beta_\vartheta)^3}$$

If a' is the accelaration experienced by the particle $\Rightarrow \beta = \beta_\vartheta \Rightarrow$

$$a' = \frac{a}{\gamma^3(1-\beta^2)^3} = \boxed{\gamma^3 a = a'} \quad (2);$$

that is, the accelaration of the particle with respect to the lab frame a is related to a' by Eq. (2). Here, $\gamma = (1+\beta_\vartheta^2)^{-1/2}$ is related to the velocity of the particle at the given instant (not any another frame's velocity!)

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Now, let us derive some useful identities regarding Lorentz factor.

$$\gamma^2\beta^2 = \frac{\beta^2}{1-\beta^2} + 1 - 1 = \frac{1}{1-\beta^2} - 1 = \gamma^2 - 1$$

$$\Rightarrow \boxed{\gamma\beta = \sqrt{\gamma^2-1}} \qquad (3)$$

$$\frac{d}{dt}(\gamma\beta) = \frac{d}{dt}\left(\frac{\beta}{\sqrt{1-\beta^2}}\right) = \dot{\gamma}\beta + \gamma\dot{\beta}$$

$$\dot{\gamma} = -\frac{1}{2}\gamma^3 \cdot (-2\beta)\cdot\dot{\beta} = \gamma^3\beta\dot{\beta} \Rightarrow$$

$$(\gamma\beta)^{\cdot} = \gamma\dot{\beta}\big(\underbrace{\gamma^2\beta^2 + 1}_{\gamma^2}\big) = \gamma^3\dot{\beta} \Rightarrow$$

(from Eq. (3))

$$\dot{\beta} = a/c \Rightarrow \quad \gamma^3 a = c\,\frac{d}{dt}(\gamma\beta) = a' \Rightarrow$$

$$\boxed{\frac{d}{dt}(\gamma\beta) = a'/c} \qquad (4)$$

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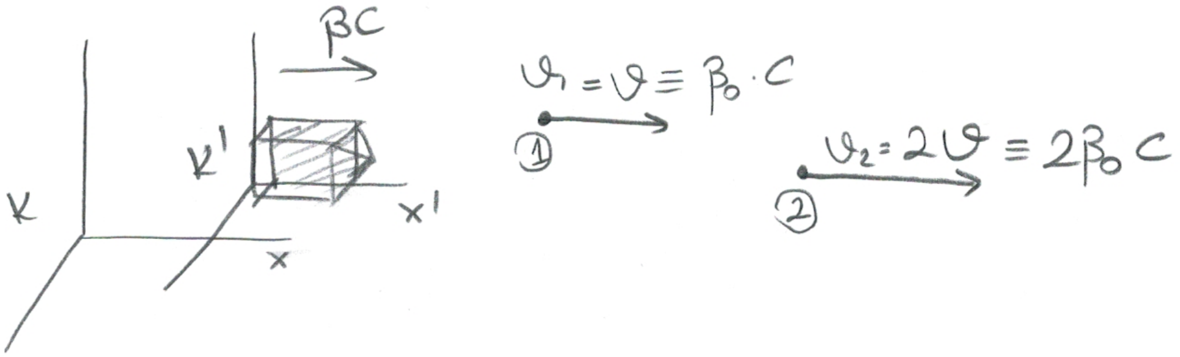
We can start solving the problem.



For the sake of clarity, let us explain the ref. frames we would be working with:

K - the frame where the spacerocket was in rest when the bullets were fired

K' - the frame where the spacerocket is in rest (at each instant, a new K' is being chosen - in this way K' is an inertial frame at each instant)

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$$\left.\begin{cases} x_1(\tau) = v_1 t = \beta_1 \tau = \beta_0 \tau \\ x_2(\tau) = 2\beta_0 \tau \end{cases}\right\}$$ bullets' position.

Spacerocket: $a' = g = \text{const} \Rightarrow$

$g/_c \, dt = d(\gamma\beta) \Rightarrow \gamma\beta = \frac{g}{c} t + \cancel{\text{const}} \Rightarrow$

(From Eq. (4)) $\qquad (\beta_{\text{initial}} = 0)$

$\gamma\beta = \frac{gt}{c} = \frac{g\tau}{c^2} \equiv u$ $\qquad$ (we will use $\begin{cases} u \equiv g\tau/c^2 \\ du = g/_{c^2} d\tau \end{cases}$)

$\Downarrow$ (From Eq. (3))

$\gamma^2 = u^2 + 1 \Rightarrow \boxed{\gamma = \sqrt{u^2+1}}$ (5)

$\Downarrow$

$$\boxed{\beta = \frac{u}{\sqrt{1+u^2}} = \frac{dx}{d\tau} = \frac{g}{c^2} \cdot \frac{dx}{du}} \quad (6)$$

$\Downarrow$

$$\frac{g}{c^2} \int_0^x dx = \int_0^{g\tau/c^2 = u} \frac{u\,du}{\sqrt{1+u^2}} = \sqrt{1+u^2} - 1 \Rightarrow$$

$$\boxed{\begin{aligned} x_{(u)} &= \frac{c^2}{g}\left(\sqrt{1+u^2} - 1\right) \\ &= \frac{c^2}{g}\left(\sqrt{1+(g\tau/c^2)^2} - 1\right) \end{aligned}} \quad (7)$$

5

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From Eq. (7), one can easily see that this is an equation of circle, the coordinates being x and iZ:

$$\left(x + \frac{c^2}{g}\right)^2 + (iZ)^2 = \left(c^2/g\right)^2 \qquad (8)$$

Taking this into account, we could also solve this problem in Minkowski diagram. We will solve in both ways!

a) Brute-Force

b) Minkowski Diagram

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a) Brute - Force

When $u = u_{1,2}$; the spacerocket catchs bullet 1, 2 $\Rightarrow$

$$x_1(u_1) = x(u_1) \Rightarrow \frac{\beta_0 \cdot c^2}{g} \cdot u_1 = \frac{c^2}{g}\left(\sqrt{1+u_1^2} - 1\right)$$

$$1+u_1^2 = (1+\beta_0 u_1)^2 = 1+\beta_0^2 u_1^2 + 2\beta_0 u_1 \Rightarrow$$

$$u_1^2(1-\beta_0^2) = 2\beta_0 u_1 \rightarrow u_1 \neq 0 \rightarrow u_1 = \frac{2\beta_0}{1-\beta_0^2} \Rightarrow$$

$$\boxed{u_1 = \frac{2\beta_0}{1-\beta_0^2} \;;\; u_2 = \frac{4\beta_0}{1-4\beta_0^2} \quad (9a)}$$

The infinitesimal increase in proper time is given with time dilation:

$$dt' = \frac{dt}{\gamma(t)} \Rightarrow du' = \frac{du}{\gamma(u)} = \frac{du}{\sqrt{u^2+1}} \Rightarrow$$

(From Eq. (5))

7.a

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$$\Delta u' = \int_{u_1}^{u_2} \frac{du}{\sqrt{u^2+1}} = \int_{y_1}^{y_2} \frac{\cancel{ch y}\, dy}{\cancel{ch y}} = \Delta y$$

$$\left| \begin{array}{l} u \equiv sh\, y \\ du = ch\, y\, dy \\ u^2+1 = sh^2 y + 1 = ch^2 y \end{array} \right. \Nearrow \qquad \Downarrow$$

$$\Delta u' = \frac{g \Delta \tau'}{c^2} = \frac{g \Delta t'}{c} = \operatorname{arcsh}(u_2) - \operatorname{arcsh}(u_1)$$

$$sh(x) = \frac{e^x - e^{-x}}{2}$$

$$ch(x) = \frac{e^x + e^{-x}}{2} \Rightarrow \operatorname{arcsh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

$+$

$$e^x = sh\,x + ch\,x$$

$$e^x = sh\,x + \sqrt{1 + sh^2 x}$$

$$\Downarrow$$

$$\frac{g\Delta t'}{c} = \ln\left(\frac{4\beta_0/(1-4\beta_0^2) + \sqrt{1 + \frac{(4\beta_0)^2}{(1-4\beta_0^2)^2}}}{2\beta_0/(1-\beta_0^2) + \sqrt{1 + \frac{(2\beta_0)^2}{(1-\beta_0^2)^2}}} \right)$$

$$= \ln\left(\frac{4\beta_0 + \sqrt{(1-4\beta_0^2)^2 + (4\beta_0)^2}}{2\beta_0 + \sqrt{(1-\beta_0^2)^2 + (2\beta_0)^2}} \cdot \left(\frac{1-\beta_0^2}{1-4\beta_0^2}\right)\right) \Rightarrow$$

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$$\begin{cases} (1-4\beta_0^2)^2+(4\beta_0)^2 = 1-8\beta_0^2+16\beta_0^4+16\beta_0^2 \\ \qquad = (1+4\beta_0^2)^2 \\ (1-\beta_0^2)^2+(2\beta_0)^2 = 1-2\beta_0^2+\beta_0^4+4\beta_0^2 \Rightarrow \\ \qquad = (1+\beta_0^2)^2 \end{cases}$$

$$\frac{g\Delta t'}{c} = \ln\left(\frac{4\beta_0+1+4\beta_0^2}{2\beta_0+1+\beta_0^2}\cdot\frac{1-\beta_0^2}{1-4\beta_0^2}\right)$$

$$= \ln\left(\frac{(1+2\beta_0)^{\cancel{2}}}{(1+\beta_0)^{\cancel{2}}}\cdot\frac{(1-\beta_0)\cancel{(1+\beta_0)}}{(1-2\beta_0)\cancel{(1+2\beta_0)}}\right)$$

$$= \ln\left(\frac{1+2\beta_0}{1-2\beta_0}\cdot\frac{1-\beta_0}{1+\beta_0}\right) \Rightarrow$$

$$\boxed{\Delta t' = \frac{c}{g}\ln\left(\frac{c+2v}{c-2v}\cdot\frac{c-v}{c+v}\right)}$$

9.a

b) $i\tau - x$ Mikowski Diagram

Mathematical Digression:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \; ; \; \cos x = \frac{e^{ix} + e^{-ix}}{2} \Rightarrow$$

$$\operatorname{sh} x = \frac{e^{x} - e^{-x}}{2} \; ; \; \operatorname{ch} x = \frac{e^{x} + e^{-x}}{2}$$

$$tg(x) = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})} = -i \cdot \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}$$

$$tg(ix) = (-i) \cdot \frac{e^{-x} - e^{x}}{e^{-x} + e^{x}} = (-i) \cdot \frac{(-\operatorname{sh} x)}{(\operatorname{ch} x)} \Rightarrow$$

$$\boxed{tg(ix) = i \tanh(x)} \quad (9b)$$

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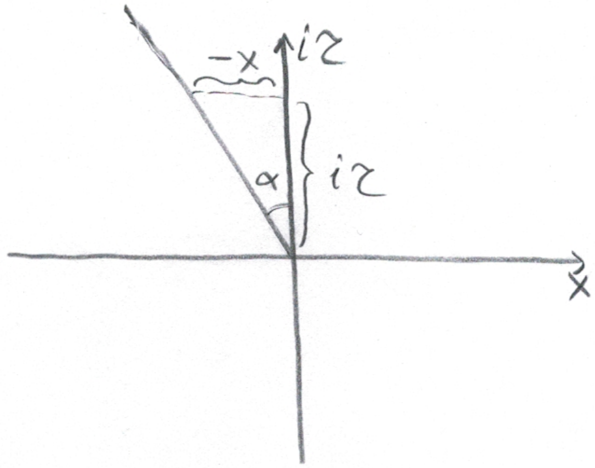
Motion with Constant Velocity in $i\tau - x$ Minkowski Diagram:

$$x_{(t)} = \vartheta t = \beta\tau = \beta\underbrace{(-i\cdot i)}_{1}\tau$$

$$x = -i\beta\cdot(i\tau)$$

$$x/i\tau = -i\beta$$

$$\operatorname{tg}\alpha = -\frac{x}{i\tau} = i\beta$$



$\Downarrow$

The motion can be drawn as a line (see figure), where

$$\operatorname{tg}\alpha = i\beta = i\tanh(\operatorname{arctanh}(\beta))$$

Using Eq. (9b):

$$\boxed{\alpha = i\cdot\operatorname{arctanh}(\beta)} \quad (10b)$$

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Mathematical Digression:

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} \Rightarrow$$

$$e^{2x} = \frac{1 + \tanh(x)}{1 - \tanh(x)} \Rightarrow$$

$$\boxed{\operatorname{arctanh}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)} \quad (11b)$$

Using this; Eq. (10b) becomes:

$$\boxed{\alpha = i \cdot \frac{1}{2} \ln\left(\frac{1+\beta}{1-\beta}\right)} \quad (12b)$$

$$\Downarrow$$

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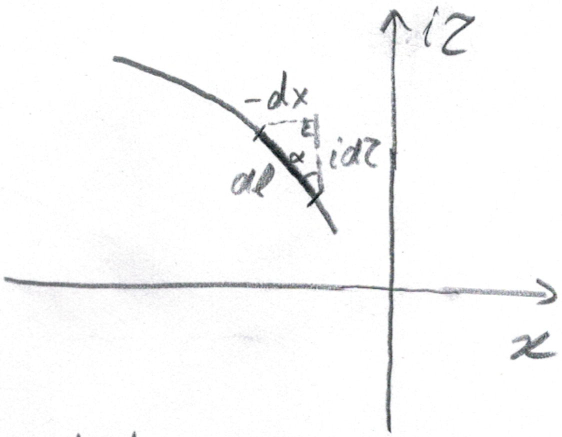
Let us generalize the concept:

Time experienced by a particle in one-dimensional motion using $i\tau - x$ Minkowski Diagram

$$\begin{cases} dx' = \gamma_{(\tau)}(dx - \beta_{(\tau)} d\tau) \\ d\tau' = \gamma_{(\tau)}(d\tau - \beta_{(\tau)} dx) \end{cases}$$



In particle's rest frame, $dx' = 0$.

Hence, the motion can be described with the curve

$$\boxed{\frac{dx}{d\tau} = \beta_{(\tau)}} \quad (13b) \quad \Rightarrow \quad \frac{dx}{d(i\tau)} = -i\beta_{(\tau)} = -tg\alpha$$

where α is determined by Eq. (10b) or Eq. (12b):

$$\begin{cases} \alpha_{(\tau)} = i \cdot \operatorname{arctanh}(\beta_{(\tau)}) \\ tg\alpha_{(\tau)} = i\beta_{(\tau)} \end{cases}$$

$$\Longrightarrow$$

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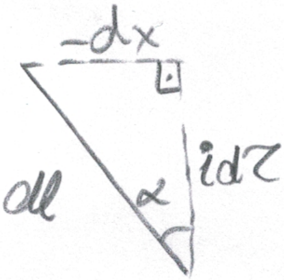
Using Eq. (13b), the time experienced by the particle is:

$$d\tau' = \gamma(d\tau - \beta^2 d\tau) = \gamma(1-\beta^2)d\tau = \frac{d\tau}{\gamma}$$

(As we expected, it is the time dilation result.)

Let us exploit a simple geometric trick: (referring to the figure)



$$\begin{cases} dl = i\dfrac{d\tau}{\cos\alpha} \\ \cos\alpha = \dfrac{1}{\sqrt{1+tg^2\alpha}} = \dfrac{1}{\sqrt{1-\beta^2}} \end{cases}$$

$$\boxed{\cos\alpha = \gamma}$$

$$dl = i \cdot \frac{d\tau}{\gamma} = i \cdot d\tau' \Rightarrow \boxed{\Delta\tau' = -i\,\Delta l} \quad (14b)$$

This is an interesting result: the time experienced by a particle is a complex multiple of the path length it traveled in the $i\tau - x$ Minkowski diagram!

11.b

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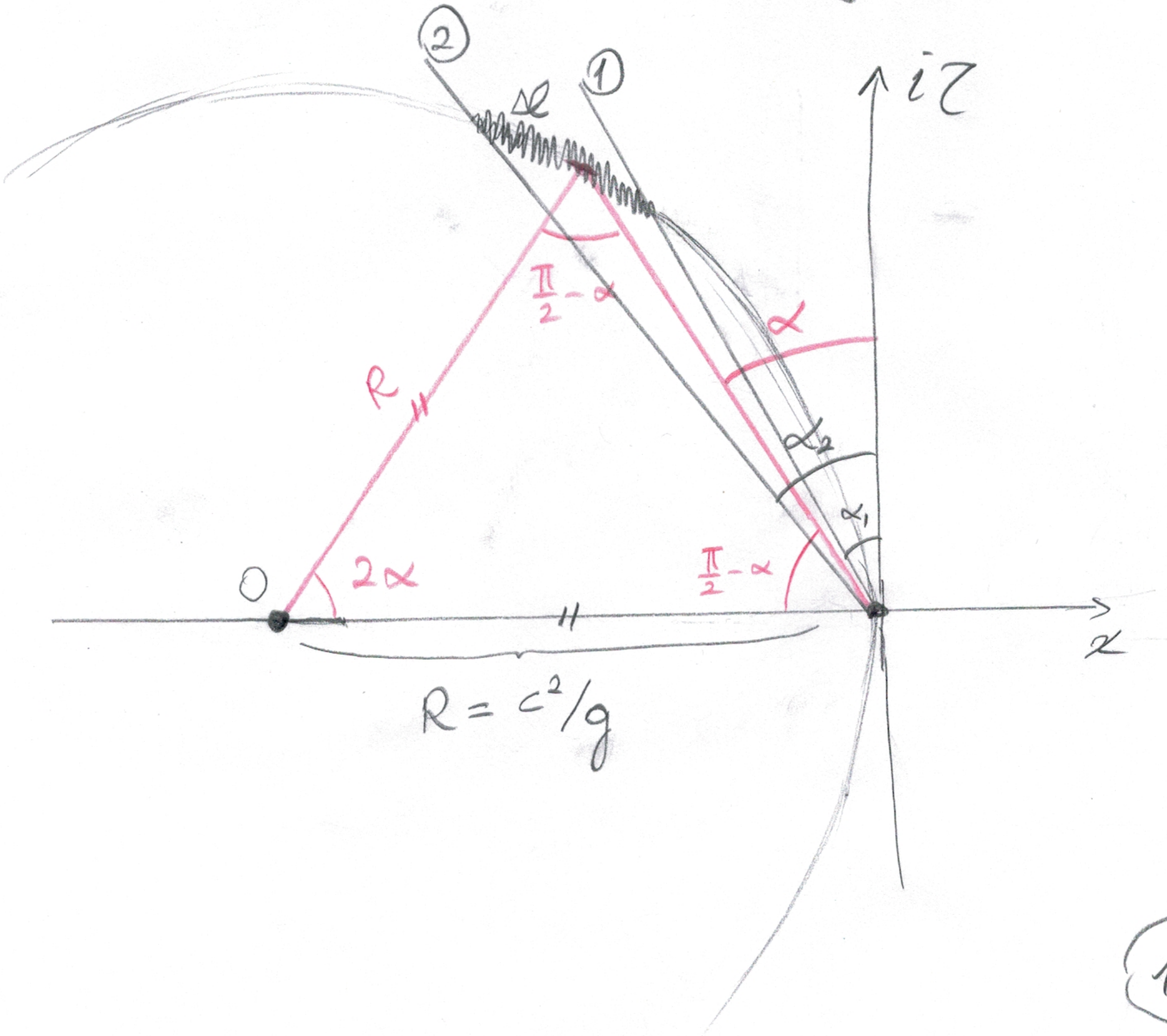
Coming back to the original question, the path followed by the space-rocket is given by Eq. (8), and the bullets have constant velocities. Thus, the $i\tau - x$ Minkowski diagram can be drawn as the following:



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From figure, we can see that

$$\Delta \ell = 2R\,\Delta\alpha = 2R(\alpha_2 - \alpha_1)$$

Using Eq.(12b) and remembering that

$$\begin{cases} \beta_1 = \beta_0 = \vartheta/c \\ \beta_2 = 2\beta_0 = 2\vartheta/c \end{cases}$$, we find:

$$\Delta \ell = iR \ln\left(\frac{1+\beta_2}{1-\beta_2} \cdot \frac{1-\beta_1}{1+\beta_1}\right) \Rightarrow$$

$$\Delta \tau' = -i\Delta\ell = R \ln\left(\frac{1+2\beta_0}{1-2\beta_0} \cdot \frac{1-\beta_0}{1+\beta_0}\right)$$

$$c\,\Delta t' = \frac{c^2}{g} \ln\left(\frac{1+2\beta_0}{1-2\beta_0} \cdot \frac{1-\beta_0}{1+\beta_0}\right) \Rightarrow$$

$$\boxed{\Delta t' = \frac{c}{g} \ln\left(\frac{c+2\vartheta}{c-2\vartheta} \cdot \frac{c-\vartheta}{c+\vartheta}\right)}$$