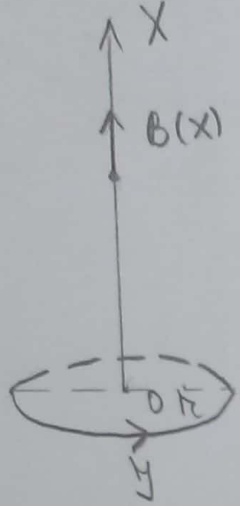
Physics Cup 2018

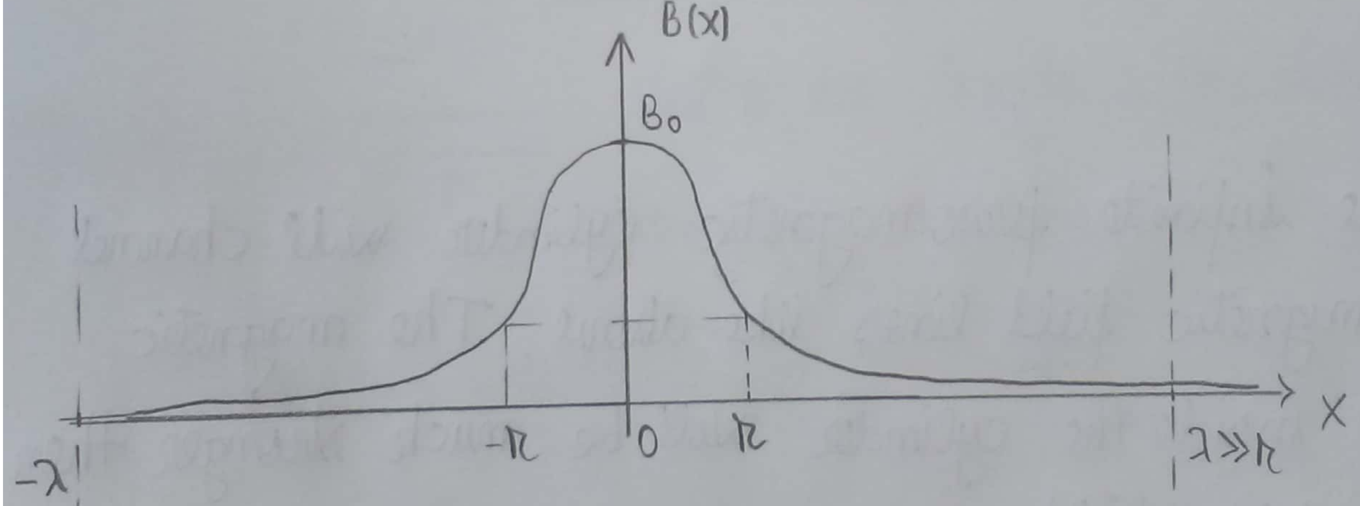
Problem 3

First, let's ignore the ferromagnetic cylinder.



From Biot-Savart law,

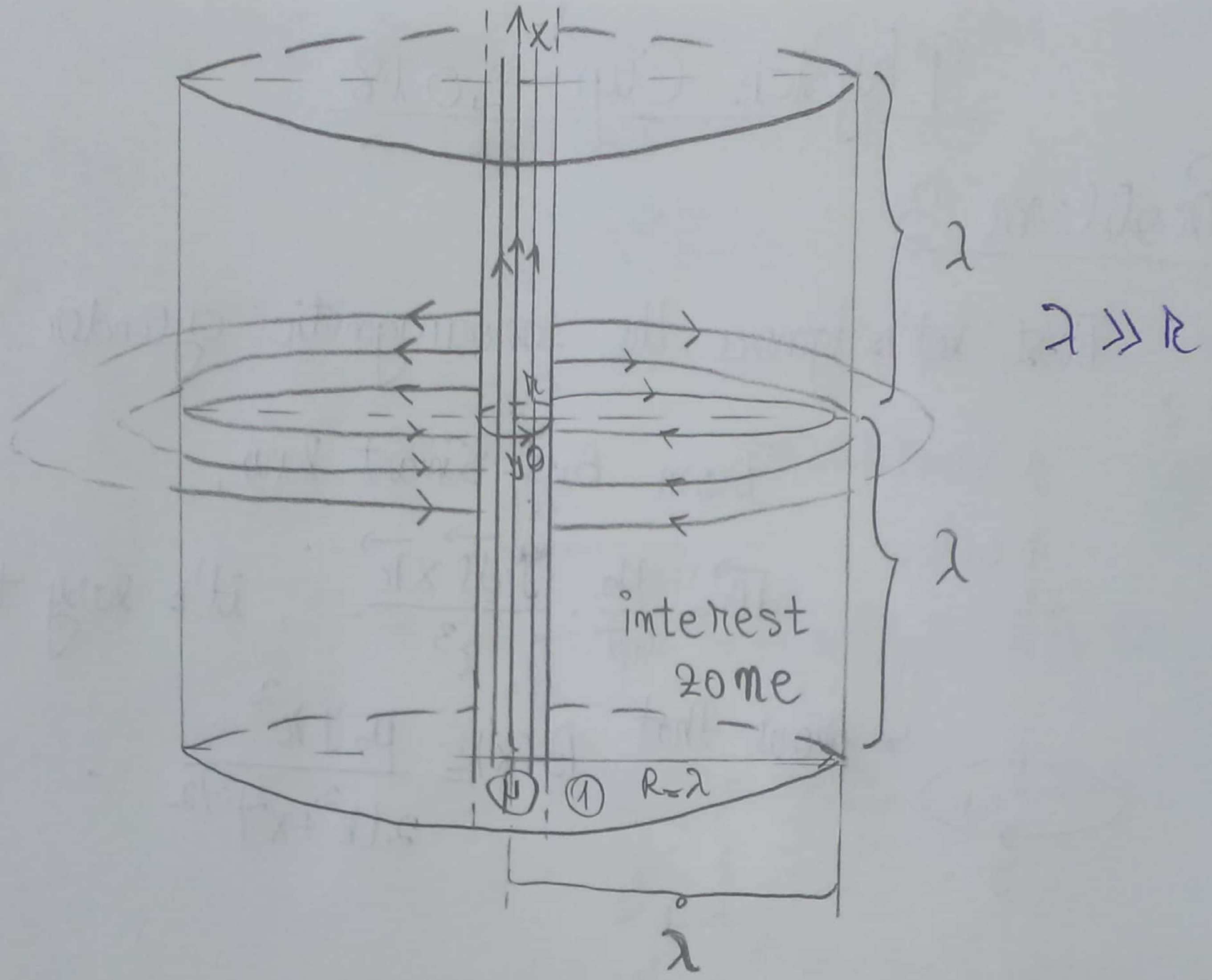
$$d\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{I\, \vec{dl} \times \vec{r}}{r^3}$$, it's easy to prove that $$B(x) = \frac{\mu_0 I R^2}{2(R^2+x^2)^{3/2}}.$$



For $x \gg R$, $B(x)$ seems to fade away.

The interest area is between $-\lambda$ and λ, with $\lambda \gg R$.

Now, we can introduce the ferromagnetic cylinder. For simplicity, let's focus only on the magnetic field content in a cylinder with $R = \lambda$ and $h = 2\lambda$.

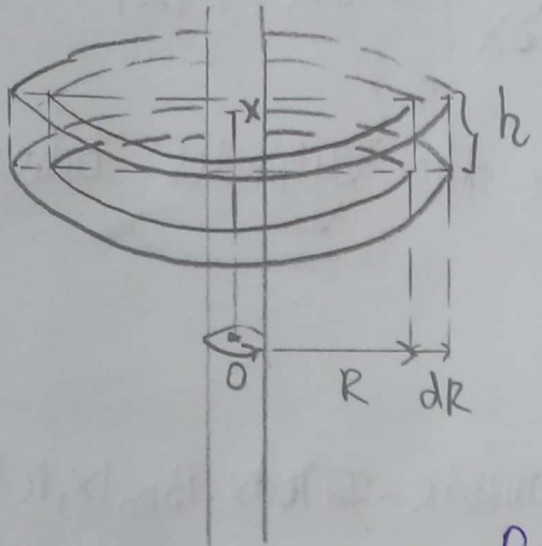


The infinite ferromagnetic cylinder will channel the magnetic field lines like above. The magnetic field inside the cylinder will be much stronger then the outside field.

Because $\mu \gg 1$, we can say that inside the cylinder, $B_{inside} \cong B_{ai}(x, R)$ - the axial component is dominant.

Outside, $B_{outside} \cong B_{ro}(x, R)$ - the radial component is dominant.

Let's use Gauss theorem for a thin cylinder of height $h \ll \lambda$ and of radius R.

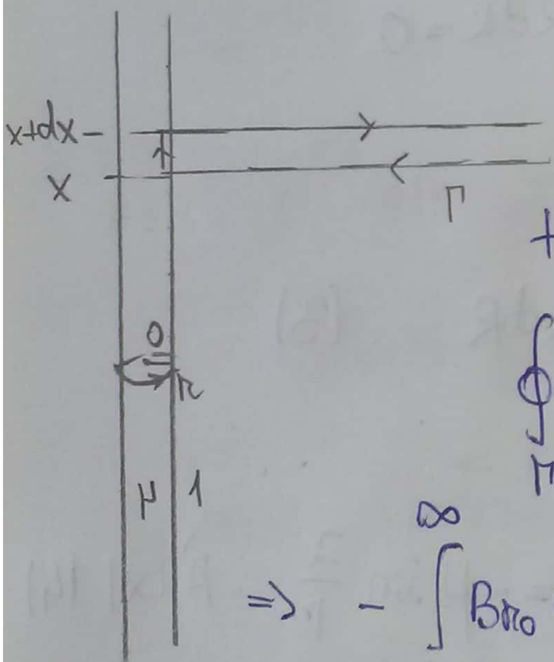


$$\nabla \cdot \vec{B} = 0$$

$$\Rightarrow B_{ro}(x,R) \cdot 2\pi R \cancel{h} = B_{ro}(x+dx, R) \cdot 2\pi(R+dR)\cancel{h} = 0$$

$$\Rightarrow d\left(B_{ro}(x,R) \cdot R\right) = 0$$

$$\Rightarrow B_{ro}(x,R) = \frac{A(x)}{R} \qquad (1)$$



Let's use Ampère's circulation theorem for the loop Γ.

$$\oint_\Gamma \vec{H} \cdot \vec{dl} = 0$$

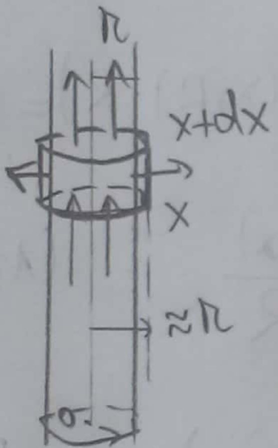
$$\Rightarrow -\int_r^\infty B_{ro}(x,R)\,dR + \frac{B_{ai}(x,r)\,dx}{\mu} + \int_r^\infty B_{ro}(x+dx,R)\,dR = 0$$

$$\Rightarrow B_{ai}(x,r) = -\mu \int_r^\infty \frac{\partial B_{ro}(x,R)}{\partial x}\,dR$$

This is a diverging integral. We can solve this by truncating it at the border of our interest zone.

$$B_{ai}(x,r) \approx -\mu \int_{r}^{\lambda} \frac{\partial B_{r0}(x,R)}{\partial x}\, dR \qquad (2)$$

Let's use Gauss law for the cylinder below.



$$\nabla \cdot \vec{B} = 0$$

$$\Rightarrow \int_0^r B_{ai}(x,R)\cdot \cancel{2\pi} R\, dR - \cancel{2\pi}\, r\, dx \cdot B_{r0}(x,r) -$$

$$- \int_0^r B_{ai}(x+dx,R)\cdot \cancel{2\pi} R\, dR = 0$$

$$\Rightarrow \quad B_{r0}(x,r) = -\frac{1}{r}\int_0^r \frac{\partial B_{ai}(x,R)}{\partial x} R\, dR \qquad (3)$$

(1) in (2): $$B_{ai}(x,r) \approx -\mu \int_r^{\lambda} A'(x)\cdot \frac{dR}{R} = -\mu \ln\frac{\lambda}{r}\cdot A'(x) \quad (4)$$

(1) in (3): $$\frac{A(x)}{r} = -\frac{1}{r}\int_0^r \frac{\partial Bai(x,R)}{\partial x} R\,dR$$

$$\Rightarrow A'(x) = -\int_0^r \frac{\partial^2 Bai(x,R)}{\partial x^2} R\,dR \qquad (5)$$

(5) in (4): $$Bai(x,r) \approx \mu \ln\frac{7}{r} \int_0^r \frac{\partial^2 Bai(x,R)}{\partial x^2} R\,dR. \qquad (6)$$

From (6), it's easy to see that $Bai(x,R) = f(x)\cdot g(R)$.

Let's say that $Bai(x,R) = C\cdot f(x)\cdot g(R)$.

From (6), $f(x) \propto f''(x) \Rightarrow f(x) = e^{-\alpha x}$, $\alpha > 0$

(6): $$\cancel{C}\cdot\cancel{e^{-\alpha x}}\cdot g(r) \approx \mu \ln\frac{7}{r}\cdot\alpha^2\,\cancel{e^{-\alpha x}}\cdot\cancel{C}\int_0^r g(R)\cdot R\,dR.$$

$$g(r) \approx \mu \ln\frac{7}{r}\cdot\alpha^2\int_0^r g(R)\cdot R\,dR.$$

To have dimensional balance,

$$\underbrace{Z\cdot g(r)\cdot r^2}_{} = Z r^2 \mu \ln\frac{7}{r}\cdot\alpha^2 \underbrace{\int_0^r g(R)\cdot R\cdot dR}_{}.$$

$$\Rightarrow \begin{cases} Z g(r)\cdot r^2 = \int_0^r g(R) R\,dR. \\ Z r^2 \mu \ln\frac{7}{r}\cdot\alpha^2 = 1 \end{cases}$$, where Z is adimensional

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An intuitive solution for $Z\, g(R) \cdot R^2 = \int_0^R g(R) \cdot R\, dR$

is $g(R) = \Delta = \text{const}$.

$$\Rightarrow Z \cdot \Delta \cdot R^2 = \Delta \cdot \frac{R^2}{2} \Rightarrow Z = \frac{1}{2}$$

$$\Rightarrow \alpha = \frac{1}{R}\sqrt{\frac{2}{\mu \ln \frac{\lambda}{R}}}$$

So, $B_{ai}(x, R) = \underbrace{C \cdot \Delta}_{E} \cdot e^{-\alpha x}$.

$$B_{ai} = B_{ai}(x) = E\, e^{-\alpha x} \qquad (7)$$

Let's write Ampère's circulation theorem for a loop extending along the axis of the ferromagnetic to infinity (and closing into a closed loop at infinity).

$$\int_{-\infty}^{\infty} H_{ai}(x)\, dx = I$$

$$\rightarrow \int_{-\infty}^{\infty} B_{ai}(x)\, dx = \mu_0 \mu I$$

Up to this point, all the equations were true for $x > 0$.

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$x<0$ doesn't bring any problem, because of the symmetry $[B_{ai}(x) = B_{ai}(-x)]$.

So,

$$2\int_0^\infty B_{ai}(x)\,dx = \mu_0\mu I$$

$$2\epsilon\int_0^\infty e^{-\alpha x}\,dx = \mu_0\mu I$$

$$-\frac{2\epsilon}{\alpha}(0-1) = \mu_0\mu I \Rightarrow \epsilon = \frac{\mu_0\mu I\alpha}{2}$$

Now, $B_{ai}(x) \approx \frac{\mu_0\mu I}{2}\alpha e^{-\alpha x}$

The magnetic flux through the loop is:

$$\phi_0 = B_{ai}(0)\cdot\pi r^2$$

$$\left.\begin{aligned} \phi_0 &= \frac{\mu_0\mu\alpha\pi r^2}{2}\cdot I \\ \phi_0 &= L\cdot I \end{aligned}\right\} \rightarrow L = \frac{\mu_0\mu\alpha\pi r^2}{2}$$

By substituting α, $L = \frac{\mu_0\mu\pi r^2}{2}\cdot\frac{1}{r}\sqrt{\frac{2}{\mu\ln\frac{z}{r}}}$

$$L \approx \mu_0\pi r\sqrt{\frac{\mu}{2\ln\frac{z}{r}}}$$

To get rid of λ, we'll use the estimation of our interest zone: $e^{-\alpha\lambda} \approx e^{-1}$

$$\Rightarrow \alpha\lambda \approx 1 \Rightarrow \frac{\lambda}{R}\sqrt{\frac{2}{\mu \ln \frac{\lambda}{R}}} \approx 1$$

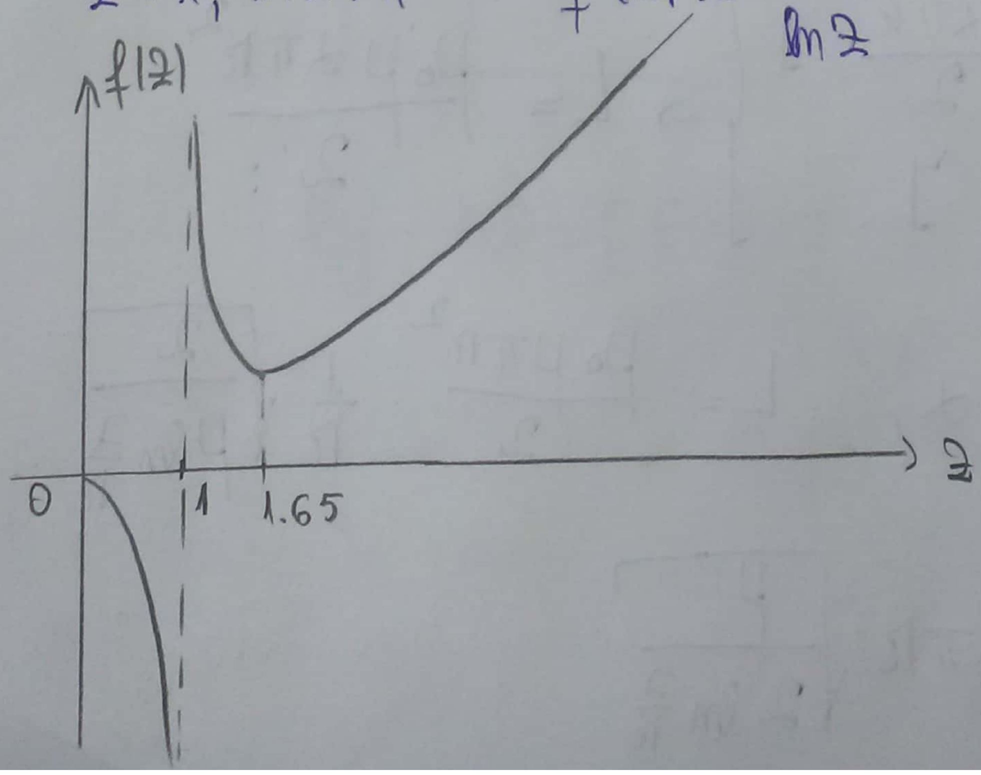
$$z \equiv \frac{\lambda}{R} \gg 1 \; ; \qquad z\sqrt{\frac{2}{\mu \ln z}} \approx 1$$

$$\Rightarrow \frac{\ln z}{z^2} \approx \frac{2}{\mu} \qquad ; \; z \gg 1$$

$$f(z) = \frac{z^2}{\ln z} \; ; \; f(z_0) \approx \frac{\mu}{2}$$

$$f'(z) = \frac{z(2\ln z - 1)}{\ln^2 z} = \frac{z}{\ln z}\left(2 - \frac{1}{\ln z}\right)$$

$$z \gg 1 ; \ln z \gg 1 \rightarrow f'(z) \approx \frac{2z}{\ln z}$$



The exact value of z_0 it's impossible to get without any numerical data for μ, but an estimation can be made.

For $z \gg 1$, $\frac{z^2}{\ln z} \approx z^2$ (order of magnitude)

$$\Rightarrow z_0 \approx \sqrt{\frac{2}{\mu}}^{-1} \; ; \; z_0 \approx \sqrt{\frac{\mu}{2}} \; ; \; z_0 \propto \sqrt{\mu}$$

$$\Rightarrow \boxed{L \approx \mu_0 \pi r \sqrt{\frac{\mu}{\ln \frac{\mu}{2}}}}$$

$$\ln \frac{\mu}{2} = \ln \mu - \ln 2 \approx \ln \mu$$

Finally, $\boxed{L \approx \mu_0 \pi r \sqrt{\frac{\mu}{\ln \mu}}}$

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