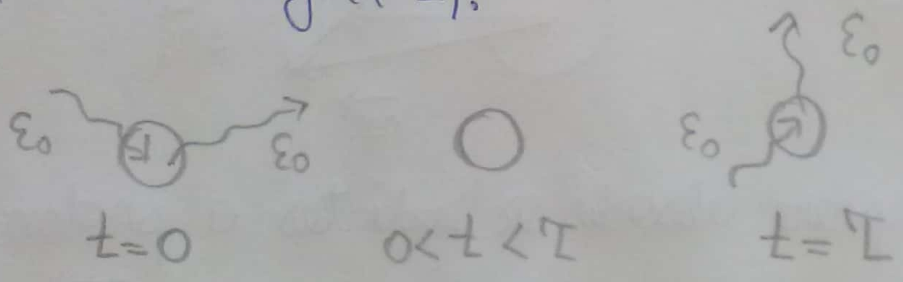
Physics Cup 2018

Problem 5

In our model, the sphere moves with $v_0 \ll c$. Also, we don't consider the rotational movement of the body.

The sphere absorbs and radiates electromagnetic radiation with $P_{abs} = P_{rad} = 4\pi R^2 \sigma T^4 \equiv P$.

At short time scale, we can consider that the body absorbs and radiates a typical photon (ε_0) at a typical time length (τ).



Plank' law as a function of frequency ν:

$$u_\nu(\nu,T) = \frac{2h\nu^3}{c^2}\cdot\frac{1}{e^{\frac{h\nu}{k_BT}}-1}$$

$$\left(\frac{\partial u_\nu}{\partial \nu}\right)_{\nu=\nu_0} = 0$$ (the major component of radiation)

Scanned by CamScanner

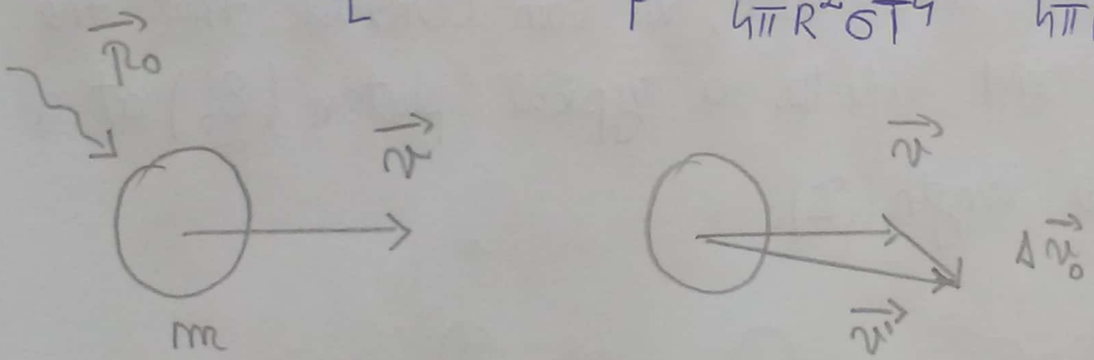
$$\Rightarrow \ -\frac{h\nu_0}{k_B T}\cdot\frac{e^{\frac{h\nu_0}{k_B T}}}{e^{\frac{h\nu_0}{k_B T}}-1}+3=0$$

$$x\equiv\frac{h\nu_0}{k_B T}\ ; \quad (x-3)e^{x}+3=0$$

Using Newton's method, we can find $x\approx 2{,}8214$.

$$\Rightarrow \ \mathcal{E}_0=h\nu_0=x\,k_B T$$

$$P=\frac{\mathcal{E}_0}{\tau}\Rightarrow \tau=\frac{\mathcal{E}_0}{P}=\frac{x\,k_B T}{4\pi R^2\sigma T^4}=\frac{x\,k_B}{4\pi R^2\sigma T^3}.$$



When the sphere absorbs or radiates a photon, the velocity changes with a typical value $\Delta\vec{v}_0$.

$$\vec{p}_0+\vec{p}_{in}=\vec{p}_{fin}\Rightarrow \Delta\vec{p}=\vec{p}_0 \text{ (absorbtion)}$$

$$m\Delta\vec{v}_0=\vec{p}_0, \text{ for absorbtion.}$$

Obviously, $m\Delta\vec{v}_0=-\vec{p}_0$, for emission.

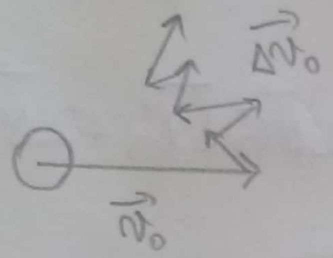
In both cases, $\Delta v_0=\frac{p_0}{m}=\frac{\mathcal{E}_0}{mc}$.

Scanned by CamScanner

The black sphere is a Brownian particle with $i = 3$ degree of freedom.

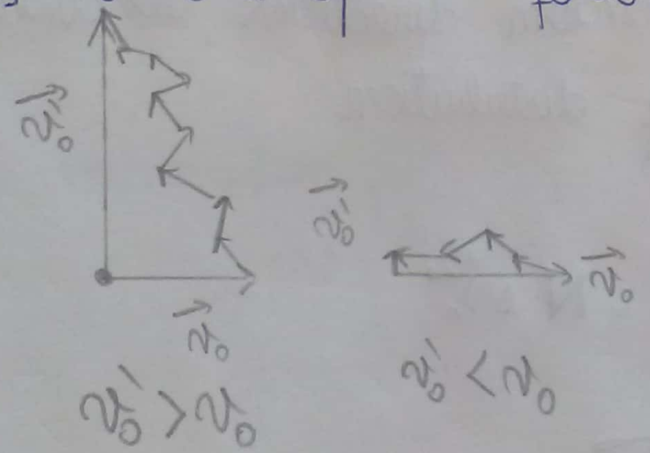
$$\frac{m v_0^2}{2} = \frac{3}{2} k_B T \rightarrow v_0 = \sqrt{\frac{3 k_B T}{m}}$$

The sphere's velocity performs a random walk.



(Projection on a plane)

Turning the velocity vector of the sphere by $\frac{\pi}{2}$ leads to one of the following situation.



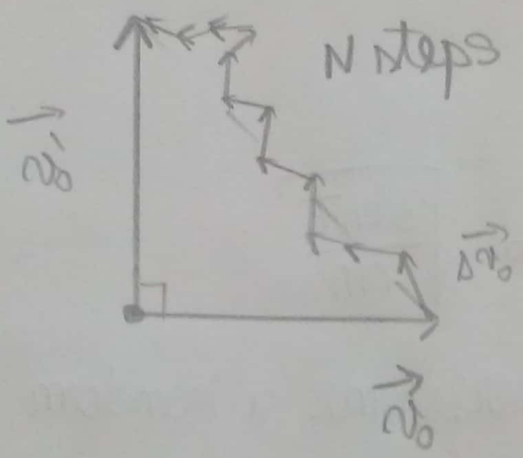
(Projection on a plane)

v_0' strives to be as close as possible to v_0, because the body is in thermal equilibrium at T.

$\Rightarrow$ An average random walk leads to $v_0' = v_0$.

Scanned by CamScanner

$$|\Delta\vec{v}| = v_0\sqrt{2}$$



$$\vec{\Delta v} = \sum_{j=1}^{N} \Delta\vec{v}^{*}_{0j}$$

$$|\Delta\vec{v}|^2 = \left(\sum_{j=1}^{N} \Delta\vec{v}^{*}_{0j}\right)\cdot\left(\sum_{j=1}^{N} \Delta\vec{v}^{*}_{0j}\right)$$

$$|\Delta\vec{v}|^2 = \sum_{j=1}^{N} |\Delta\vec{v}^{*}_{0j}|^2 + \underbrace{2\,\Delta\vec{v}^{*}_{01}\cdot\Delta\vec{v}^{*}_{02} + 2\,\Delta\vec{v}^{*}_{01}\cdot\Delta\vec{v}^{*}_{03} + 2\,\Delta\vec{v}^{*}_{02}\cdot\Delta\vec{v}^{*}_{03} + \dots}$$

very large number of scalar products of pairs of vectors pointing in random directions, all drawn from the same probability distribution

$$\Rightarrow \quad |\Delta\vec{v}|^2 = \sum_{j=1}^{N} |\Delta\vec{v}^{*}_{0j}|^2 = N\,\Delta v_0^{*2}$$

$$\Rightarrow \quad |\Delta\vec{v}| = \Delta v_0^{*}\sqrt{N}$$

$$\Rightarrow \quad \Delta v_0^{*}\sqrt{N} = v_0\sqrt{2} \rightarrow N = 2\,\frac{v_0^2}{\Delta v_0^{*2}}$$

Δv_0^{*} - average velocity shift given by an absorbtion and an emission (one immediately after the other).

Scanned by CamScanner

$$\overrightarrow{\Delta\nu_0^*} = \overrightarrow{\Delta\nu_0}^{abs} + \overrightarrow{\Delta\nu_0}^{em} \quad ; \quad \Delta\nu_0^{abs} = \Delta\nu_0^{em} = \Delta\nu_0$$

$$|\overrightarrow{\Delta\nu_0^*}|^2 = 2\,\Delta\nu_0^2 + \langle \overrightarrow{\Delta\nu_0}^{abs} \cdot \overrightarrow{\Delta\nu_0}^{em} \rangle$$

$$\langle \overrightarrow{\Delta\nu_0}^{abs} \cdot \overrightarrow{\Delta\nu_0}^{em} \rangle = \Delta\nu_0^2 \langle \cos\theta \rangle = 0 \quad \text{(isotropy)}$$

$$\Rightarrow \quad \Delta\nu_0^* = \Delta\nu_0 \sqrt{2}$$

$$\Rightarrow \quad N = 2\,\frac{\nu_0^2}{2\,\Delta\nu_0^2} = \frac{\nu_0^2}{\Delta\nu_0^2}$$

The total average turning time is:

$$\tau^{total} = N\tau = \frac{\nu_0^2}{\Delta\nu_0^2} \cdot \frac{x k_B}{4\pi R^2 \sigma T^3}$$

The mean free path is:

$$\lambda = \nu_0 \cdot \tau^{total} = \frac{\nu_0^3}{\Delta\nu_0^2} \cdot \frac{x k_B}{4\pi R^2 \sigma T^3}$$

$$\lambda = \frac{3^{3/2}}{4\pi \cdot x} \cdot \frac{\sqrt{m k_B}\, c^2}{\sigma R^2} \cdot T^{-7/2}$$

$$\boxed{\lambda \approx 0.1466\, \frac{\sqrt{m k_B}\, c^2}{\sigma R^2}\, T^{-7/2}}$$

by DOLTEANU ȘTEFAN

Scanned by CamScanner