Physics Cup - Problem 2
Geometric Solution

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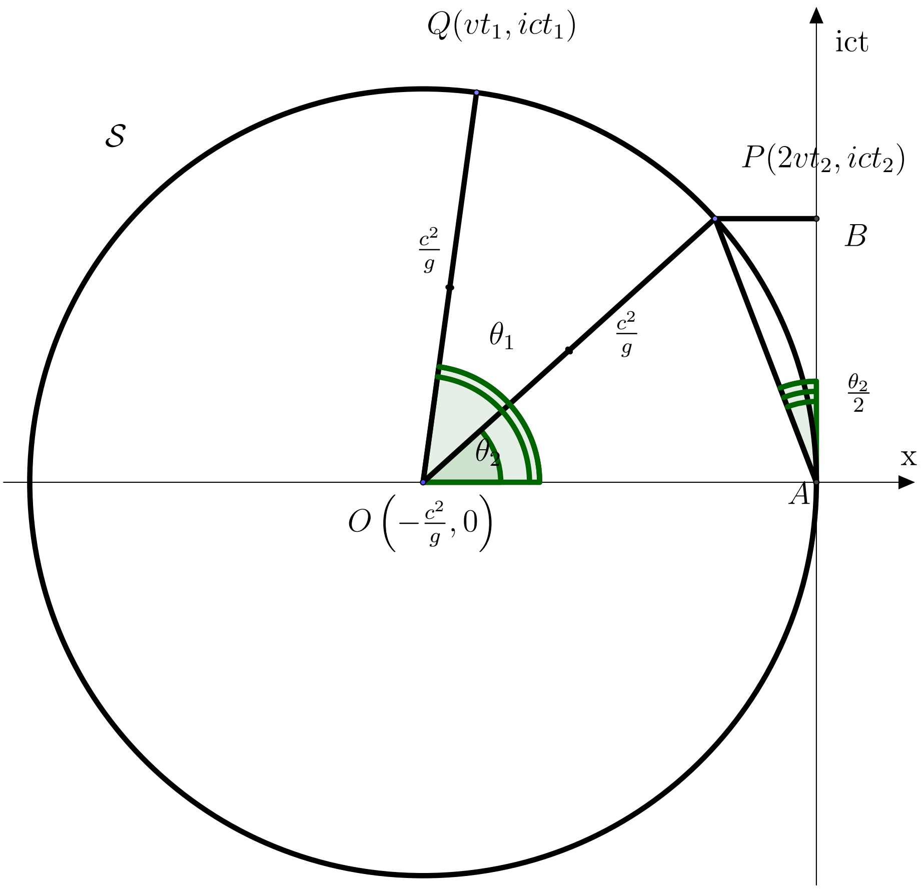
We will make use of the fact that in a suitable reference frame the position of the spaceship can be described by the following hyperbola:

$$\left(x+\frac{c^2}{g}\right)^2-(ct)^2=\frac{c^4}{g^2}$$

This can be written in the following form, where i is the imaginary unit:

$$\left(x+\frac{c^2}{g}\right)^2+(ict)^2=\frac{c^4}{g^2}$$

This is indeed the equation of a circle in the x-ict diagram, with radius $\frac{c^2}{g}$ and centre $\left(-\frac{c^2}{g},0\right)$:



Here $\mathcal{S}$ is the world line of the spaceship, P and Q are the events when the faster and slower missiles are caught respectively, O is the centre of $\mathcal{S}$, A is the origin of the diagram, B is the orthogonal projection of P to axis ict, $\theta_2 = POA\angle$ and $\theta_1 = QOA\angle$.
It is clear that $P = (2vt_2, ict_2)$ for some t_2 as the faster missile moves with constant speed $2v$. To obtain the proper time we have to find the length of the arc PQ which is simply:

$$ic\tau = \frac{c^2}{g}(\theta_1 - \theta_2) \tag{1}$$

Triangle POA is isosceles, so $PAO\angle = 90^\circ - \frac{\theta_2}{2}$, thus:

$$PAB\angle = \frac{\theta_2}{2}$$

From the right triangle PAB:

$$\tan\frac{\theta_2}{2} = \frac{PB}{AB} = \frac{2vt_2}{ict_2} = \frac{2v}{ic} = -i\frac{2v}{c} \tag{2}$$

Using Euler's identity:

$$\tan x = \frac{\sin x}{\cos x} = \frac{e^{ix} - e^{-ix}}{2i} \cdot \frac{2}{e^{ix} + e^{-ix}} = \frac{1}{i} \cdot \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} = -i\tanh(ix) \tag{3}$$

Substituting (3) into (2):

$$-i\frac{2v}{c} = -i\tanh\frac{i\theta_2}{2}$$

This yields:

$$\theta_2 = -2i\tanh^{-1}\left(\frac{2v}{c}\right) \tag{4}$$

With the same train of thoughts:

$$\theta_1 = -2i\tanh^{-1}\left(\frac{v}{c}\right) \tag{5}$$

Substituting (4) and (5) into (1):

$$ic\tau = \frac{c^2}{g}\left(-2i\tanh^{-1}\left(\frac{v}{c}\right) + 2i\tanh^{-1}\left(\frac{2v}{c}\right)\right)$$

From this:

$$\tau = \frac{2c}{g}\left(\tanh^{-1}\left(\frac{2v}{c}\right) - \tanh^{-1}\left(\frac{v}{c}\right)\right)$$