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In the spaceship's rest frame, according to Newton's 2nd law, we have $F_s = mg$. Transforming this force into the lab frame, and using the fact that parallel forces do not alter in different frames, we obtain: $F_{lab} = mg$. Using that $F_{lab} = \frac{dp}{dt}$, where p is momentum and t time measured by *the lab frame*, we get $\frac{dp}{dt} = mg$. Integrating this equation, we obtain equation (1) for the momentum p:

$$p = mgt \qquad (1)$$

Using the definition of relativistic momentum $= \frac{mv}{\sqrt{1-(v/c)^2}}$, substituting equation (1) and isolating v, we obtain equation (2):

$$v(t) = \frac{gt}{\sqrt{1+(gt/c)^2}} \qquad (2)$$

In order to find displacement $x(t)$ measured by the lab frame we integrate equation (2), obtaining equation (3):

$$x(t) = \frac{c^2}{g}(\sqrt{1+(gt/c)^2} - 1) \quad (3)$$

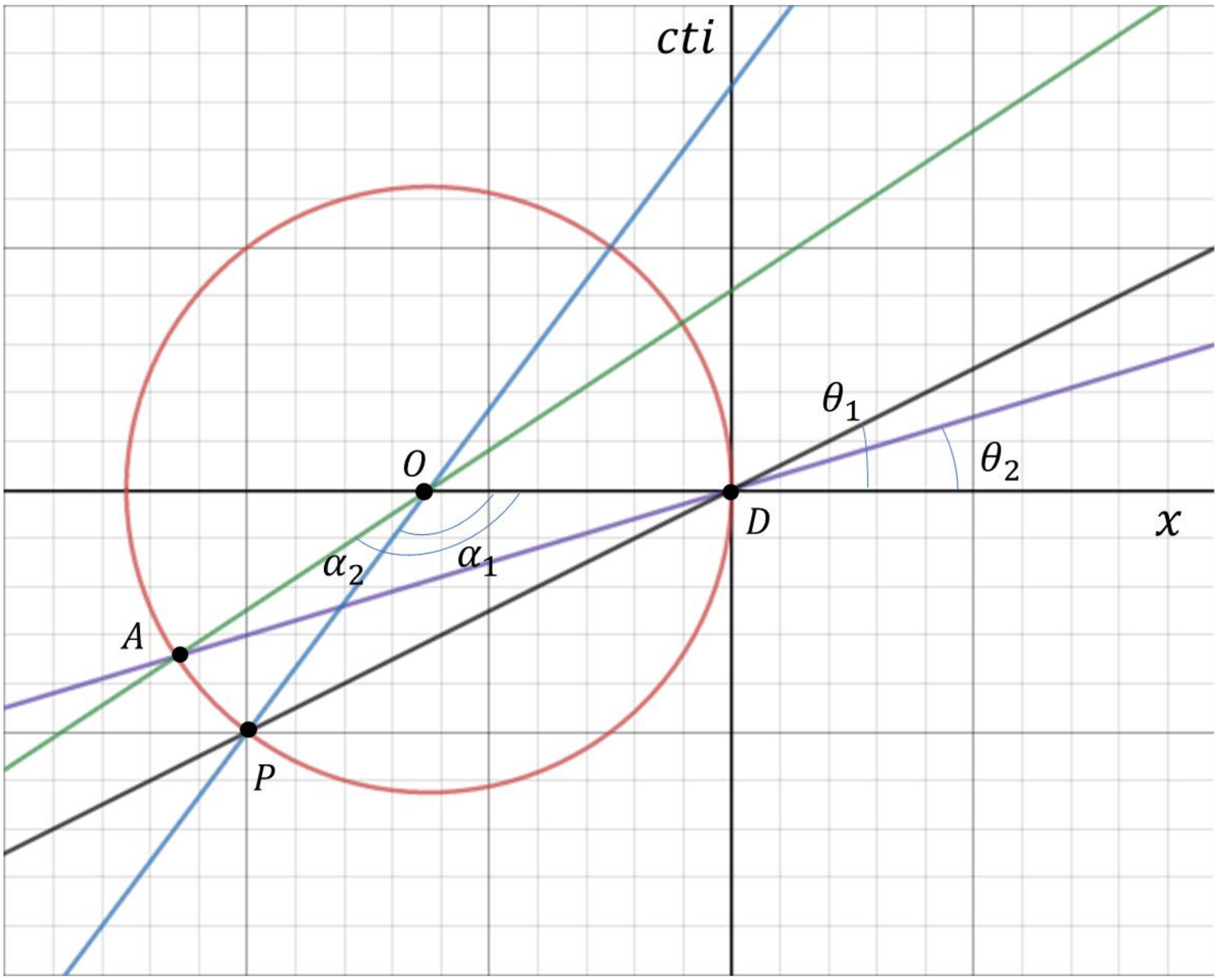
Rearranging equation (3) we obtain equation (4) which defines a hyperbola as below:

$$\left(\frac{gx}{c^2}+1\right)^2 - \left(\frac{gt}{c}\right)^2 = 1 \qquad (4)$$

Defining y as $y = cti$, isolating $t = \frac{y}{ci}$ and substituting into equation (4) we get:

$$\left(\frac{gx}{c^2}+1\right)^2 + \left(\frac{gy}{c^2}\right)^2 = 1 \qquad (5)$$

Equation (7) defines a circle with radius $R = \frac{c^2}{g}$, and center coordinates $(x_c, y_c) = (-\frac{c^2}{g}, 0)$. Using this equation we can plot the graph below, where O is the center of the circle. We also have that $\alpha_1 = \pi - 2\theta_1$ and also $\alpha_2 = \pi - 2\theta_2$.



Let dS be the length of an infinitesimal arc with height dy and width dx, such that $dS = \sqrt{(dx)^2 + (dy)^2}$. Since $y = cti$, we have $dy = cidt$, which substituted into dS, results in: $dS = \sqrt{(dx)^2 - (cdt)^2} = i\sqrt{(cdt)^2 - (dx)^2} = -icd\tau$, where $cd\tau = \sqrt{(cdt)^2 - (dx)^2}$.

Using $dS = icd\tau$, and integrating the equation, we obtain equation (6) where $\Delta\alpha = \alpha_2 - \alpha_1$:

$$\int_{S_0}^{S} dS = -ic\int_{\tau_1}^{\tau_2} d\tau \Leftrightarrow R\Delta\alpha = -ic\Delta\tau \ (6)$$

In the figure above we have the motion of the first and second rocket in the y_1 and y_2 lines respectively. Since their motion is given by $x_1 = vt$ and $x_2 = 2vt$, and that $y = cti \Leftrightarrow t = \frac{y}{ci}$, therefore the lines have equations:

$$y_1 = \frac{ci}{v}x \qquad (7)$$

$$y_2 = \frac{ci}{2v}x \qquad (8)$$

The angles θ_1 and θ_2 formed between the x axis and the y_1 and y_2 lines are given by

$\theta_1 = \arctan\left(\frac{ci}{v}\right)$ and $\theta_2 = \arctan\left(\frac{ci}{2v}\right)$ respectively.

Using that $\tan(\theta) = \tan\left(\frac{\pi-\alpha}{2}\right) = \frac{1}{\tan\left(\frac{\alpha}{2}\right)}$, which implies that $\tan\left(\frac{\alpha_1}{2}\right) = -\frac{iv}{c}$ and also $\tan\left(\frac{\alpha_2}{2}\right) = -\frac{2iv}{c}$, and also the mathematical definition $\arctan(ix) = i\mathrm{arctanh}(x)$ we obtain $\alpha_1 = -2iarctanh(\frac{v}{c})$ and also $\alpha_2 = -2iarctanh(\frac{2v}{c})$.

Now using equation (6) we obtain:

$$\frac{2ic^2}{g}\left(\mathrm{arctanh}\left(\frac{2v}{c}\right) - \mathrm{arctanh}\left(\frac{v}{c}\right)\right) = ic\Delta\tau \Leftrightarrow \Delta\tau = \frac{2c}{g}\left(\mathrm{arctanh}\left(\frac{2v}{c}\right) - \mathrm{arctanh}\left(\frac{v}{c}\right)\right) \quad (9)$$

Thus the final result is:

$$\Delta\tau = \frac{2c}{g}(\mathrm{arctanh}\left(\frac{2v}{c}\right) - \mathrm{arctanh}\left(\frac{v}{c}\right))$$