PC Problem 2

Bonus Attempt,Prathyush P

Following the hints, let us look at the $x - ict$ diagram. We'll need to remember the following points:

- $\tanh\theta = -i\tan i\theta$
- In the $x - ict$ diagram, it is the euclidean norm $ds^2 = dx^2 + (icdt)^2$which is conserved, so the Lorentz transformation is basically the rotation of the frame by an (imaginary) angle θ
- τ is the proper time of the rocket

Let us first find ϕ, the angle between the rest frame and the frame of moving particle in terms of it's velocity. Let the particle be moving with velocity v (fig1).Using simple geometry, we can show that:

$$\tan\phi = \frac{dx}{icdt} = \frac{-iv}{c} \Rightarrow i\tan\phi = \frac{v}{c} \text{———————Eq 1}$$

Let us go to the instantaneous inertial frame of the rocket . In a time $d\tau$, the frame rotates by angle $d\phi$ when it gains velocity $gd\tau$. Thus,

$$\tan d\phi = d\phi = \frac{-igd\tau}{c}$$

$$\implies i\phi = \frac{g\tau}{c}$$

Thus, the angle ϕ increases uniformly with time.
Using the above point, and the fact that the rocket moves along the time axis in it's instantaneous inertial frame, we can deduce that the path of the rocket is a circle as seen from the rest frame(fig 2).

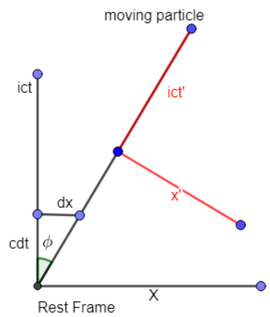

fig1

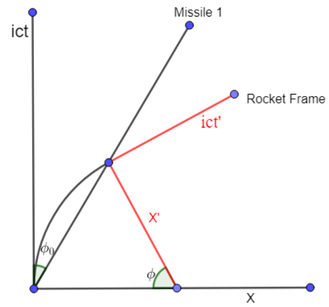

fig2

When the rocket intercepts the 1st missile, let it's frame have rotated by an angle ϕ. Let the missile's frame be rotated by an angle ϕ_0. Here, the rest frame is the instantaneous inertial frame of the rocket at the time of launching of missiles. From fig 2, we can deduce that $\phi = 2\phi_0$.
If we make the substitution $\phi_0 \longrightarrow -i\phi_0$, eq 1 becomes

$$-i\tan(i\phi_0) = \tanh\phi_0 = \frac{v}{c}$$

$$\implies \phi_0 = \tanh^{-1}(\frac{v}{c})$$

Using the same substitution, we get $\phi = \frac{g\tau}{c}$. Since the relationship between ϕ and ϕ_0 still holds under the substituion, we get

$$\frac{g\tau}{c} = 2\tanh^{-1}(\frac{v}{c})$$

$$\implies \tau = \frac{2c}{g}\tanh^{-1}(\frac{v}{c})$$

Thus, the proper time interval between the rockets catching the 2 missile is

$$\Delta\tau = \frac{2c}{g}(\tanh^{-1}(\frac{2v}{c}) - \tanh^{-1}(\frac{v}{c}))$$