# P-5 Solution

Let us consider the case when the magnitude of the velocity of the ball is much smaller than the speed of light. A photon absorbed by the sphere can thus be assumed to be scattered isotropically. Let the wave vector of the photon be $\vec{k}$, and the velocity vector of the sphere be $\vec{v}$. Then momentum conservation gives

$$\boxed{m \Delta \vec{v} = \frac{h \vec{k}}{2\pi}}$$

In the velocity space, since $\hat{k}$ is random, we have a random walk of step length $\frac{kh}{2\pi m}$.

Since the velocity vector rotates by $90^\circ$, the "displacement" in velocity space is $\sqrt{2}\, v_0$, where $v_0$ is the velocity of the sphere. We thus have

$$\sqrt{N_s}\,\sqrt{\langle \Delta v^2 \rangle} = \sqrt{2}\, v_0 \Rightarrow N_s = \frac{8\pi m^2 v_0^2}{h^2 \langle k^2 \rangle} \quad ⊛$$

where $N_s$ = number of scattering steps, and the average is done over the number of photons. The path length can then be given by

$$N_s v_0 \Delta t = \frac{8\pi^2 m^2 v_0^3 (\Delta t)}{h^2 \langle k^2 \rangle}$$

where $\Delta t$ is the time between two collisions of a photon and the sphere.

Treating the sphere as if it were a wall, if $N^*$ is the number density of photons, the collision frequency is

$$\nu = \frac{1}{4} \pi R^2 c N^*$$

⊛ See the remarks for a justification.

Since photons have integral quantised $|\vec{L}|$*, they follow Bose statistics. Hence, their distribution is

$$dn = \frac{V}{\pi^2 c^3} \frac{\omega^2 \, d\omega}{\exp(\hbar\omega / k_B T) - 1}$$

So, $$N^* V = \frac{V}{\pi^2 c^3} \int_0^\infty \frac{\omega^2 \, d\omega}{\exp(\hbar\omega / k_B T) - 1}$$

$$\hbar\omega / k_B T = x$$

$$\Rightarrow \quad N^* = \frac{1}{\pi^2 c^3} \int_0^\infty \frac{k_B^3 T^3}{\hbar^3} \frac{x^2 \, dx}{e^x - 1}$$

$$= \frac{8\pi^3 k_B^3 T^3}{\pi^2 h^3 c^3} \int_0^\infty \frac{x^2 \, dx}{e^x - 1}$$

$$= \frac{8\pi k_B T}{h^3 c^3} \int_0^\infty \frac{x^2 e^{-x} \, dx}{1 - e^{-x}}$$

$$\int_0^\infty \frac{x^2 e^{-x} \, dx}{1 - e^{-x}} = \int_0^\infty \left( x^2 e^{-x} + x^2 e^{-2x} + x^2 e^{-3x} + \cdots \right) dx$$

$$= \int_0^\infty x^2 e^{-x} \, dx + \int_0^\infty \frac{(2x)^2}{2^2} e^{-2x} \frac{d(2x)}{2} + \cdots$$

$$= \left[ 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \right] \int_0^\infty x^2 e^{-x} \, dx$$

$$= \zeta(3) \, \Gamma(3) = 2 \zeta(3)$$

Where $\zeta$ is the Riemann zeta function, & $\Gamma$ is the gamma function. $[\Gamma(n) = (n-1)! \quad \forall n \in \mathbb{N}]$

* Reduced angular momentum in units with $\hbar = 1$ etc

So $N^* = \dfrac{16\pi k_B^3 T^3 \zeta(3)}{\hbar^3 c^3}$

$$\Rightarrow \nu = \frac{4\pi^2 \zeta(3) R^2 k_B^3 T^3}{\hbar^3 c^2}$$

$$\Rightarrow \Delta t = \frac{1}{\nu} = \boxed{\frac{\hbar^3 c^2}{4\pi^2 \zeta(3) R^2 k_B^3 T^3}}$$

Now, $k = \omega/c$, so

$$\langle k^2 \rangle_{\text{all photons}} = \frac{\langle \omega^2 \rangle_{\text{all photons}}}{c^2}$$

$$= \frac{1}{c^2} \cdot \int \frac{\omega^2 \, dn}{\int dn}$$

$$= \frac{1}{c^2 N^* V} \int_0^\infty \frac{V \omega^4 \, d\omega}{\pi^2 c^3 \left[e^{\hbar\omega/k_BT} - 1\right]}$$

$$= \frac{\hbar^3 c^3 V}{c^2 V \cdot 16\pi k_B^3 T^3 \zeta(3) \pi^2 c^3} \int_0^\infty \frac{\omega^4 \, d\omega}{e^{(\hbar\omega/k_BT)} - 1}$$

$$= \frac{\hbar^3}{16\pi^3 \zeta(3) c^2 k_B^3 T^3} \int_0^\infty \frac{x^4 \, dx}{e^x - 1} \cdot \left(\frac{2\pi k_B T}{h}\right)^5$$

(#)

$$= \frac{2 k_B^2 T^2 \pi^2}{\zeta(3) \hbar^2 c^2} \cdot \zeta(5) \cdot \Gamma(5)$$

$$= \boxed{\frac{48\zeta(5)\pi^2}{\zeta(3)} \; \frac{k_B^2 T^2}{\hbar^2 c^2}}$$

(#) This integral is completely analogous to the $\zeta(3)\Gamma(3)$ integral.

So path length = $l$, where

$$l = \frac{\cancel{8\pi} m^2 v_0^3 \times \dfrac{h^3 c^2}{\cancel{4\zeta(3)\pi} k_B^3 T^3}}{24 \cancel{h^2} \dfrac{\cancel{48}\zeta(5)}{\cancel{\zeta(3)}} \pi^2 \dfrac{k_B^2 T^2}{\cancel{h^2} c^2} \pi R^2}$$

$$= \boxed{\frac{m^2 v_0^3 h^3 c^4}{24\zeta(5) \pi^3 k_B^5 T^5 R^2}}$$

Now since the sphere is in thermal equilibrium with the radiation, we can estimate $v_0$ using equipartition of energy, to get

$$\frac{1}{2} m v_0^2 = \frac{3}{2} k_B T$$

$$\Rightarrow \boxed{v_0 = \sqrt{\frac{3k_B T}{m}}}$$

So,

$$l = \frac{m^2 h^3 c^4}{24\zeta(5)\pi^3 k_B^5 T^5 R^2} \frac{3\sqrt{3} \cdot k_B^{3/2} T^{3/2}}{m^{3/2}}$$

$$= \boxed{\frac{h^3 c^4}{8\zeta(5)\pi^3 R^2} \sqrt{\frac{3m}{(k_B T)^7}}}$$

Some remarks:

① In the calculation of $N_{s,total}$, the total number of steps, we used $\langle k^2 \rangle$. This was done due to the following reason:

The net displacement squared is reasonably indicated by variance of the displacements ~~[illegible]~~ ($x$) multiplied by the number of steps.

$$\mathrm{Var}[x]^2 = E[x^2] - E[x]^2$$
$$= E[x^2]$$

as $E[x] = 0$ for a random distribution around 0.

$$E[x^2] = \left[\sum n_{\text{collision}, k} \cdot \frac{h^2 k^2}{4\pi^2 m}\right] \Big/ \left[\sum n_{\text{collision}, k}\right]$$

Since the number of collisions of the sphere with a photon with wave number between $[k, k+dk]$ is proportional to the number of such photons, so we only need to average $\frac{h^2 k^2}{4\pi^2 m}$ over the number of photons, which establishes our claim.

[ It can also be shown by a more direct (but equivalent) argument :

$$k^2 n_k \sim \text{displacement}^2 \Rightarrow N_{s,\text{total}} \langle k^2 \rangle \sim \text{displacement}^2 \;]$$

(2) The factor $T$ occurs everywhere as a combination with $k_B$ as only $k_B T$. This illustrates the purely statistical thermodynamic definition of $T$ as a parameter characterising the energy of particles, with characteristic energy scale $k_B T$.

(3) As a check for the veracity of the result obtained, we examine some limits :

(a). $m \to 0$ or $\infty$.

If $m$ is small, we would expect it to traverse a larger path, in view of the result of the analogous Brownian motion in material media.

However, the dependence is only $\propto \sqrt{m}$, which is due to the fact that $v_0$ also increases with decreasing m. The case $m \to \infty$ is analogous.

(b) $T \to 0$.

When $T \to 0$, the average |impulse| due to the photons is quite small, which would make us believe that the sphere would travel for along ~~time~~ characteristic time. The factor $v_0$ still reduces the dependence on T, though not to a large extent.

(c). $R \to 0$ or $\infty$

As $R \to 0$, there would be much less area for the photons to strike on, so collisions will be lesser compared to larger R, which increases the characteristic time.

(4) When m, T, R are expressed in SI units,

$$l \approx 6.2 \times 10^{11} \frac{(m/1kg)^{1/2}}{(R/1m)^2 \; (T/1K)^{7/2}} \; m$$

At room temperature ($T \approx 300K$), for a 1kg ball of radius 10 cm,

$$l \approx 1.3 \times 10^3 \times \frac{(m/1kg)^{1/2}}{(R/1m)^2} \; m \approx 130 \text{ km}$$

& $v_0 \approx 1.1 \times 10^{-10}$ m/s.

So this effect won't be apparent on usual laboratory time scales. However, this effect becomes apparent for small masses.