PC Problem 2

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Some things to note:

- The ground frame is the instantaneous inertial frame of the rocket at the time of launching of missiles
- τ is the proper time of the rocket
- $v(\tau)$ is the velocity of rocket as measured by the ground frame

We shall first find $v(\tau)$. Let us go to a instantaneous inertial fram moving with rocket with velocity $v(\tau)$ in ground frame. In time $d\tau$, the rocket gains velocity $gd\tau$. Thus, velocity in ground frame is

$$v(\tau + d\tau) = \frac{v + gd\tau}{1 + \frac{vgd\tau}{c^2}} = v + gd\tau - \frac{v^2 gd\tau}{c^2}$$

$$\implies dv = gd\tau(1 - \frac{v^2}{c^2})$$

substitute $v = c\tanh(\theta)$. We get

$$d\theta = \frac{gd\tau}{c}$$

At $\tau = 0$, $\theta = 0$ (because $v = 0$). Thus,$\theta = \frac{g\tau}{c} \Rightarrow v = c\tanh(\frac{g\tau}{c})$
Here, the argument θ (called the rapidity) transforms like newtonian velocity which can be verified using the relativistic addition formula. Indeed, for small velocities, the rapidity is proportional to the velocity.

Thus, velocity of missile 1 in the rocket frame is $v = c\tanh(\phi - \frac{g\tau}{c})$, where ϕ is the rapidity of the missile. Thus, distance travelled by the missile in rocket frame is:

$$s = \int_0^\tau c\tanh(\phi - \frac{g\tau}{c})d\tau = \frac{-c^2}{g}\ln\frac{\cosh(\phi - \frac{g\tau}{c})}{\cosh\phi}$$

Thus, when the rocket catches the missile, $\frac{g\tau}{c} = 2\phi$

Thus, proper time interval between catching the 2 missiles is

$$\frac{2c}{g}(\tanh^{-1}(\frac{2v}{c}) - \tanh^{-1}(\frac{v}{c}))$$