Physics Cup Q4 Dylan Toh

Let the angle between the plates be $\phi = 2\theta$, and the volume in the vessel be V; then $V = L \cdot h \cdot (h \tan\theta) = Lh^2 \tan\theta$, where h is the height of the liquid (from bottom apex to water level surface).

We know that at equilibrium, the gravitational potential energy of the system is minimised. Since the mass of ropes and plates are negligible, thus the condition is equivalent to the lowest possible centre of mass of the liquid. Note that $COM_{water} = 2l\cos\theta - \frac{2}{3}h$ from point P (because the centre of mass of a triangular prism is a third of its height from the base). Thus the equilibrium angle θ for the given volume V is the θ when COM_{water} is maximised. We know that at maximum COM_{water}, one has $\frac{d}{d\theta}COM_{water} = 0$; thus

$$\frac{d}{d\theta}\left(2l\cos\theta - \frac{2}{3}h\right) = -2l\sin\theta - \frac{2}{3}\frac{d}{d\theta}\sqrt{\frac{V}{L\tan\theta}} = \frac{1}{3}\sqrt{\frac{V}{L}}\frac{\sec^2\theta}{\tan^{3/2}\theta} - 2l\sin\theta = 0$$

Now let the equilibrium angle specifically when the vessel is almost filled up to the rim, be $\theta = \theta_0$. Then $h_0 = l\cos\theta_0 \therefore V_0 = Ll^2\cos^2\theta_0\tan\theta_0$

$$\therefore \frac{1}{3}\sqrt{\frac{Ll^2\cos^2\theta_0\tan\theta_0}{L}}\frac{\sec^2\theta_0}{\tan^{3/2}\theta_0} - 2l\sin\theta_0 = 0 \therefore \frac{1}{3\sin\theta_0} - 2\sin\theta_0 = 0$$

$$\therefore \sin\theta_0 = \frac{1}{\sqrt{6}} \therefore \theta_0 = \arcsin\frac{1}{\sqrt{6}}$$

thus the angle between the two plates is

$$\phi_0 = 2\theta_0 = \mathbf{2\,arcsin\frac{1}{\sqrt{6}}} = \mathbf{arcsin\frac{\sqrt{5}}{3}} = \mathbf{arccos\frac{2}{3}}$$

To find the lowest-frequency oscillation, we use $\xi = \theta - \theta_0$ as the generalised coordinate. We have the potential energy

$$\Pi = -\rho g V(COM_{water}) = -\rho g V\left(2l\cos\theta - \frac{2}{3}h\right) = -\rho g V\left(2l\cos\theta - \frac{2}{3}\sqrt{\frac{V}{L\tan\theta}}\right)$$

Which we may expand around $\theta = \theta_0$ as $\Pi(\theta_0 + \xi) \cong \Pi(\theta_0) + \kappa\xi^2/2$ and evaluate

$$\kappa = \Pi''(\theta_0) = -\frac{d}{d\theta}\left(\frac{1}{3}\sqrt{\frac{V}{L}}\frac{\sec^2\theta}{\tan^{3/2}\theta} - 2l\sin\theta\right)\Bigg|_{\theta_0} \cdot \rho g V$$

$$= -\left(\frac{1}{3}\sqrt{\frac{V}{L}}\left(\frac{1}{2}\tan^{-1/2}\theta - \frac{3}{2}\tan^{-5/2}\theta\right)\sec^2\theta - 2l\cos\theta\right)\Bigg|_{\theta_0} \cdot \rho g V$$

$$= \left[-\frac{1}{3}\sqrt{\frac{Ll^2\cos^2\theta_0\tan\theta_0}{L}}\left(\frac{1}{2}\tan^{-1/2}\theta_0 - \frac{3}{2}\tan^{-5/2}\theta_0\right)\sec^2\theta_0 + 2l\cos\theta_0\right] \cdot \rho g V$$

$$= \left[-\frac{1}{3}\cos\theta_0\left(\frac{1}{2} - \frac{3}{2}\tan^{-2}\theta_0\right)\sec^2\theta_0 + 2\cos\theta_0\right] \cdot \rho g l(Ll^2\cos^2\theta_0\tan\theta_0)$$

and when we substitute in $\sin\theta_0 = 1/\sqrt{6},\ \cos\theta_0 = \sqrt{5}/\sqrt{6}, \tan\theta_0 = 1/\sqrt{5}$:

$$\kappa = \frac{2\sqrt{2}}{\sqrt{3}}\rho g L l^3$$

Finding the kinetic energy as a function of $\dot{\xi}$ is a bit trickier. We may assume (due to small oscillation) that each water molecule oscillates in a small linear path about equilibrium point (minimum action). The linear path is given by the path of a point under the instantaneous linear transformation between the water's triangle shape at $\theta = \theta_0$ to the shape at $\theta = \theta_0 + \xi$ (where $\xi = d\theta \approx 0$). We can find this linear transformation as:

- Translation upward by $-d(2l\cos\theta) = 2l\sin\theta\, d\theta = 2l\sin\theta_0 \cdot \xi$
- Scale along x-axis (take apex as origin) by factor $(1+\alpha)$ where $\alpha \approx 0$
- Scale along y-axis (take apex as origin) by factor $1/(1+\alpha) = (1-\alpha)$

Where we note that the order of these three transformations don't matter because they are on the order of $\xi = d\theta$. We find α (treat it as a differential variable) by the change in $\tan\theta$:

$$\frac{1+\alpha}{1-\alpha} = 1 + 2\alpha = \frac{\tan(\theta_0+\xi)}{\tan\theta_0} = 1 + \frac{\sec^2\theta_0}{\tan\theta_0}\xi \therefore \alpha = \frac{1}{\sin 2\theta_0}\cdot\xi$$

We take the apex at equilibrium as the origin. We note the point (x, y) shifts to point $(x', y') = \left((1+\alpha)x, (1-\alpha)y + 2l\sin\theta_0\cdot\xi\right)$

$$= \left(x + \frac{x}{\sin 2\theta_0}\cdot\xi, y + \left(2l\sin\theta_0 - \frac{y}{\sin 2\theta_0}\right)\cdot\xi\right)$$

and thus has velocity

$$v(x,y) = \frac{(x',y') - (x,y)}{dt} = \dot{\xi}\cdot\frac{1}{\xi}\left(\frac{x}{\sin 2\theta_0}\cdot\xi, \left(2l\sin\theta_0 - \frac{y}{\sin 2\theta_0}\right)\cdot\xi\right)$$
$$= \left(\frac{x}{\sin 2\theta_0}, 2l\sin\theta_0 - \frac{y}{\sin 2\theta_0}\right)\cdot\dot{\xi}$$

We can thus express the kinetic energy $K \cong \mu\dot{\xi}^2/2$ as

$$K = \iiint \frac{1}{2}\rho v^2(x,y)$$
$$= \frac{\rho L}{2}\int_0^{h_0}\int_{-\left(\frac{V_0}{Lh_0}\right)\frac{y}{h_0}}^{\left(\frac{V_0}{Lh_0}\right)\frac{y}{h_0}} \left[\left(\frac{x}{\sin 2\theta_0}\right)^2 + \left(2l\sin\theta_0 - \frac{y}{\sin 2\theta_0}\right)^2\right]\dot{\xi}^2\, dx\, dy$$
$$= \frac{\rho L\dot{\xi}^2}{2}\int_0^{h_0}\int_{-\frac{V_0 y}{Lh_0^2}}^{\frac{V_0 y}{Lh_0^2}} \left(\frac{x^2}{\sin^2\phi_0} + 4l^2\sin^2\theta_0 - 2ly\sec\theta_0 + \frac{y^2}{\sin^2\phi_0}\right) dx\, dy$$
$$= \frac{\rho L\dot{\xi}^2}{2}\int_0^{h_0}\left[\frac{x^3}{3\sin^2\phi_0} + x\left(4l^2\sin^2\theta_0 - 2ly\sec\theta_0 + \frac{y^2}{\sin^2\phi_0}\right)\right]_{-\frac{V_0 y}{Lh_0^2}}^{\frac{V_0 y}{Lh_0^2}} dy$$
$$= \rho L\dot{\xi}^2\int_0^{h_0}\left(\frac{V_0^3 y^3}{3L^3h_0^6\sin^2\phi_0} + \frac{V_0 y}{Lh_0^2}\left(4l^2\sin^2\theta_0 - 2ly\sec\theta_0 + \frac{y^2}{\sin^2\phi_0}\right)\right) dy$$
$$= \rho\dot{\xi}^2\left[\frac{V_0^3 y^4}{12L^2h_0^6\sin^2\phi_0} + \frac{V_0}{h_0^2}\left(2l^2\sin^2\theta_0\, y^2 - \frac{2l\sec\theta_0\, y^3}{3} + \frac{y^4}{4\sin^2\phi_0}\right)\right]_0^{h_0}$$

$$= \rho\dot{\xi}^2\left[\frac{V_0^3}{12L^2h_0^2\sin^2\phi_0} + V_0\left(2l^2\sin^2\theta_0 - \frac{2l\sec\theta_0\, h_0}{3} + \frac{h_0^2}{4\sin^2\phi_0}\right)\right]$$

sub. $V_0 = Ll^2\cos^2\theta_0\tan\theta_0 = Ll^2\sin\theta_0\cos\theta_0 = Ll^2\sin\phi_0/2$:

$$K = \rho\dot{\xi}^2\left[\frac{L^3l^6\sin^3\phi_0/8}{12L^2h_0^2\sin^2\phi_0} + \left(2l^2\sin^2\theta_0 - \frac{2l\sec\theta_0\, h_0}{3} + \frac{h_0^2}{4\sin^2\phi_0}\right)Ll^2\sin\phi_0/2\right]$$

$$= \frac{\rho\dot{\xi}^2Ll^2}{2}\left[\frac{l^4\sin\phi_0}{48h_0^2} + \left(2l^2\sin^2\theta_0 - \frac{2l\sec\theta_0\, h_0}{3} + \frac{h_0^2}{4\sin^2\phi_0}\right)\sin\phi_0\right]$$

sub. $h_0 = l\cos\theta_0$:

$$K = \frac{\rho\dot{\xi}^2Ll^2}{2}\left[\frac{l^2\sin\phi_0}{48\cos^2\theta_0} + \left(2l^2\sin^2\theta_0 - \frac{2l^2}{3} + \frac{l^2\cos^2\theta_0}{4\sin^2\phi_0}\right)\sin\phi_0\right]$$

$$= \frac{\rho\dot{\xi}^2Ll^4\sin\phi_0}{2}\left[\frac{1}{48\cos^2\theta_0} + 2\sin^2\theta_0 - \frac{2}{3} + \frac{1}{16\sin^2\theta_0}\right]$$

sub. $\sin\theta_0 = 1/\sqrt{6},\ \cos\theta_0 = \sqrt{5}/\sqrt{6}, \sin\phi_0 = \sqrt{5}/3$:

$$K = \frac{\rho\dot{\xi}^2Ll^4}{2}\cdot\frac{1}{9\sqrt{5}} \therefore \mu = \frac{1}{9\sqrt{5}}\rho Ll^4$$

We thus have angular frequency of this lowest-frequency mirror-symmetric oscillation mode as

$$\omega = \sqrt{\frac{\kappa}{\mu}} = \sqrt{\frac{\frac{2\sqrt{2}}{\sqrt{3}}\rho gLl^3}{\frac{1}{9\sqrt{5}}\rho Ll^4}} = \sqrt[4]{\mathbf{1080}}\cdot\sqrt{\frac{\boldsymbol{g}}{\boldsymbol{L}}} \approx \mathbf{5.733}\sqrt{\frac{\boldsymbol{g}}{\boldsymbol{L}}}$$