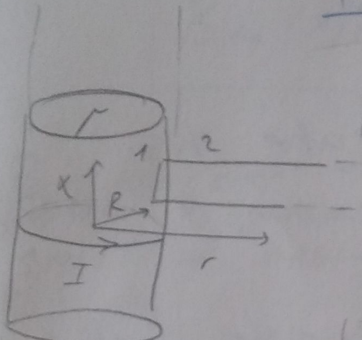
Problem 3 Yunus Emre Parmaksız



From azimuthal symmetry $B_\phi = 0$

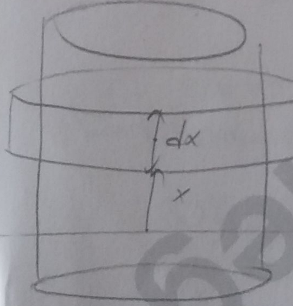
$$B_{x_1}(x)\,dx \cdot \frac{}{r} + \int_R^{\alpha R} \left(B_{r_2}(x+dx) - B_{r_2}(x)\right) dr = 0$$

$$\boxed{\frac{B_{x_1}(x,R)}{r} + \frac{d}{dx}\int_R^{\alpha R} B_{r_2}\, dr = 0} \quad (1)$$

Second equation can be obtained by writing down Gauss's Law for a cylinder of heigh dx

$$\oint \overline{B}\, \overline{dS} = 0$$

$$\int_0^R \left(B_{x_1}(x+dx) - B_{x_1}(x)\right) d(\pi r^2) = -2\pi r\, dx\, B_{r_2}$$



$$\boxed{\frac{d}{dx}\int_0^R B_{x_1}\, r\, dr = -r\, B_{r_2}} \quad (2)$$

When subsitutting equation (2) I assumed flux from outsite B_{x_2} is negligible compared to flux of B_{x_1}. This assumation holds only if αR is of order $\mu^{\beta} R$ where $\beta < 1/2$

Because B_{x_1} is of order $\frac{1}{\mu} B_{x_2}$ and

$$\phi_1 \propto B_{x_1}(\alpha R)^2 = B_{x_1} R^2 \mu^{2\beta} \ll B_{x_2} R^2$$

$r^{2\beta} << r \rightarrow 2\beta < 1$

$\beta < 1/2$

At the limit situation I will take this coefficient as 1/2. Thus equation (1) and (2) becomes:

$$\boxed{\frac{B_{x1}(x,R)}{r} = \frac{-d}{dz}\int_R^{r^{1/2}R} B_{r2}(x,r)\,dr} \quad (3)$$

$$\boxed{B_{r2}(x,r) = \frac{-1}{r}\frac{d}{dz}\int_0^R B_{x1}(x,r)\,r\,dr} \quad (4)$$

Now let us solve these two equations:

$$\frac{B_{x1}(x,R)}{r} = \left[\frac{d^2}{dz^2}\int_0^R B_{x1}(x,r)\,r\,dr\right](\ln r^{1/2})$$

By assuming seperation of variables method for $B_{x1}(x,r) \longrightarrow B_{x1}(x,r) = A_1(x)\,A_2(r)$

$$\frac{1}{\lambda^2}A_1 = \frac{d^2 A_1}{dx^2}$$ (here λ is some constant coefficient)

$A_1 = Fe^{x/\lambda} + Ee^{-x/\lambda}$. Since A_1 should decrease with increasing x our function should be of form: $A_1 = Ee^{-x/\lambda}$

$$\boxed{B_{x_1}(x,r) = B_{x_1}(0,r)\, e^{-z/\lambda}} \quad (5)$$

By inserting eq (5) into eq (3) we find a simillior result for B_{r_2}

$$\frac{B_{x_1}(0,R)}{\Gamma}\, e^{-x/\lambda} = \frac{-d}{dz}\int_{R}^{\Gamma^{1/2}R} B_{r_2}(x,r)\, dr$$

Since $B_{r_2}(x,r)$ is of form $B_{r_2}(x,r) = D(x)\,\frac{A}{r}$

we obtain:

$$\frac{B_{x_1}(0,R)}{\Gamma}\, e^{-x/\lambda} = -\frac{dD}{dx}\, A \ln(\Gamma^{1/2})$$

$D = D_0\, e^{-x/\lambda} \longrightarrow$ So we finally found the form of $B_{r_2} \rightarrow$

$$\boxed{B_{r_2}(x,r) = \frac{A}{r}\, e^{-x/\lambda}} \quad (6)$$

To find the dependence of B_{x_1} to r I will use Ampere's circulation theorem for another loop shown in the figure (1).

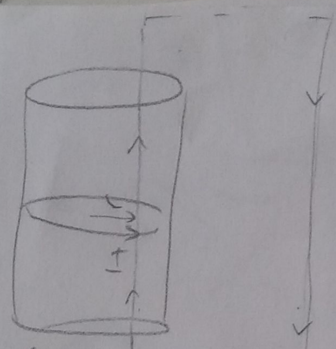


figure (1)

$$\oint \vec{H}\, d\vec{l} = I$$

radial magnetic field will give no contribution since it is at $x \to \infty$.

Thus: $B_{x1}(0,r) \int_0^{\infty} e^{-x/\lambda}\, dx = \frac{\mu\mu_0 I}{2}$

$$B_{x1}(0,r) = \frac{\mu\mu_0 I}{2\lambda}$$

$$\boxed{B_{x1}(x,r) = \frac{\mu\mu_0 I}{2\lambda} e^{-x/\lambda}} \quad (7)$$

$$\boxed{B_{r2}(x,r) = \frac{A}{r} e^{-x/\lambda}} \quad (8)$$

From equation (1)

$$\frac{1}{r}\left(\frac{\mu\mu_0 I}{2\lambda}\right) = \frac{1}{\lambda} A \ln(r^{1/2})$$

$$A = \frac{\mu_0 I}{\ln(\mu)}$$

Now that we find $B_{r2}(x,r)$ completely we can calculate the coefficient λ using equation (2)

$$\left(\frac{\mu\mu_0 I}{2\lambda}\right)\frac{1}{\lambda}\frac{R^2}{2} = A = \frac{\mu_0 I}{\ln\mu}$$

$$\lambda^2 = \mu\ln\mu\,\frac{R^2}{4} \longrightarrow \lambda = \frac{R}{2}\sqrt{\mu\ln\mu}$$

Finally let's calculate the flux at $(x=0)$.

$$\phi = \int_0^R B_{x1}(x,r)\,2\pi r\,dr\Big|_{x=0}$$

$$= \frac{\mu\mu_0 I}{2\lambda}\cdot\pi R^2 = \frac{\mu\mu_0 I\pi R^2}{2\cdot\frac{R}{2}\sqrt{\mu\ln\mu}}$$

$$\phi = \mu_0\pi R\cdot I\sqrt{\frac{\mu}{\ln\mu}}$$

From the definition of inductance $L = \frac{\phi}{I}$ we find it as:

$$\boxed{L = \mu_0\pi R\sqrt{\frac{\mu}{\ln\mu}}}$$