Problem 1

Let us write continuity equation

$$\nabla \cdot (\rho \vec{v}) + \frac{\partial \rho}{\partial t} = 0$$

Density is constant since the liquid is incompressible ($\rho = const$) Hence

$$\rho \, \nabla \cdot \bar{v} = 0 \rightarrow \nabla \cdot \bar{v} = 0 \quad (1)$$

And the liquid is said to be vortex-free, which gives us the condition

$$\oint \bar{v} \, d\bar{l} = \int (\nabla \times \bar{v}) \, d\bar{S} = 0 \rightarrow \nabla \times \bar{v} = 0 \quad (2)$$

everywhere

Eq (1) and (2) is simillier to that of an Elektric Field equations. We shall define a potential for the velocity of the liquid:

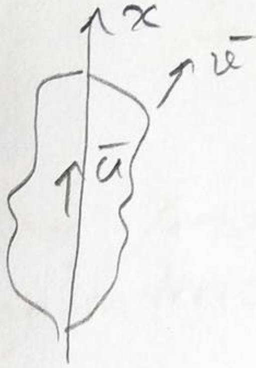
$$\bar{v} = \nabla \psi \quad (3)$$

which satisfies the Laplace's equation

$$\nabla^2 \psi = 0 \quad (4)$$

Our aim is to calculate the kinetic energy of liquid, hence the integral:

$$I = \int v^2 dV \quad (5)$$



Without knowing the shape of our body (boundary conditions) solving (3) and (4) is impossible.

Therefore we will use the polarization constant that has been given specifically for our body. To do that we have to establish a connection between $\vec{v}$ and $\vec{E}$ in two distinct systems. Let us write boundary conditions for two fields:

Velocity

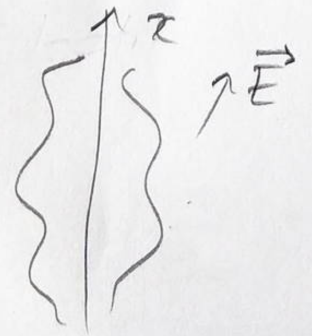
$\vec{v}\vec{n} = \vec{u}\vec{n}$ (surface) (6)

$\vec{v} \to 0$ (infinity) (7)

Electric Field

$\vec{E} = 0$ (inside) (8)

$\vec{E} \to \vec{E_0}$ (infinity) (9)



Electric field consists of two fields: $\vec{E} = \vec{E_0} + \vec{E_p}$

$\vec{E_0}$ is the homogeneous field

$\vec{E_p}$ is the field created by induced charges on the metallic body.

Calculating the dipol moment of the system

$$\vec{p} = \oint_{S_0 \text{ (surface of our body)}} \sigma \vec{r}\, dS = \oint \sigma (\bar{\rho} + \vec{x})\, dS$$

$$= \oint \sigma \bar{\rho}\, dS + \oint \sigma \vec{x}\, dS$$

The first integral is zero by virtue of cylindrical symmetry:

$$\vec{p} = \bar{e}_x \oint \sigma x\, dS = \alpha \vec{E_0}$$
$$= \alpha E_0 \bar{e}_x$$

Writing Gauss Law at the surface

$$\frac{\sigma}{\varepsilon_0} = \vec{E}\vec{n}$$

$$\frac{\alpha E_0}{\varepsilon_0} = \oint x \bar{E}\bar{n}\, dS = \oint x \vec{E}\, d\bar{S}$$

$$= \oint x (\vec{E_0} + \vec{E_p})\, dS$$

$$= \vec{E_0} \oint x\, d\bar{S} + \oint x \vec{E_p}\, dS$$

$$= \bar{E_0} \int \nabla x\, dV + \oint x \bar{E_p}\, d\bar{S}$$

$$= E_0 V + \oint x \bar{E_p}\, d\bar{S}$$

$$\left(\frac{\alpha}{\varepsilon_0} - V\right) E_0 = \oint_{S_0} x \vec{E_p}\, d\vec{S} \quad (10)$$

The fact that if our body is made of a homogeneous dielectric material the field inside would be homogeneous implies that the surface charge is induced in a way that $\bar{E}$ field satisfies the relation below at the surface: $\vec{E}\,\bar{n} = \bar{P}\,\bar{n}$. Here $\bar{P}$ is the constant polarization vector. Using this relation, we can write $\vec{E_p}\,\bar{n} = \bar{A}\,\bar{n}$ at the surface. $\bar{A}$ is to be found from the integral (10):

$$\left(\frac{\alpha}{\varepsilon_0} - V\right)E_0 = \oint x\,\bar{E_p}\,d\bar{S} = \oint x\bar{A}\,d\bar{S}$$
$$= \vec{A}\oint x\,d\bar{S}$$
$$= \bar{A}\,\bar{e_x}\,V$$

From symmetry A should be along x.

Hence $\vec{A}\,\vec{e_x} = A \longrightarrow A = \left(\frac{\alpha}{\varepsilon_0 V} - 1\right)E_0$ (11)

With that being said boundary conditions for velocity and electric field becomes very much alike.

<u>velocity</u>	<u>Electric Field of Induced Charges</u>
$\bar{v}\,\bar{n} = \bar{u}\,\bar{n}$ (surface)	$\overline{E_p}\,\bar{n} = \bar{A}\,\bar{n}$ (surface)
$\bar{v} \to 0$ (infinity)	$\overline{E_p} \to 0$ (infinity)

Now that everything is correlated, the required integral (5) can be calculated in electric field, and then transformed to velocity field.

$$I' = \int E_p^2 \, dV = \int \nabla\psi \nabla\psi \, dV$$
$$= \int (\nabla(\psi\nabla\psi) - \psi\nabla^2\psi) \, dV$$

$\nabla^2\psi = 0$ everywhere

$$I' = \int \nabla(\psi\nabla\psi) \, dV = \oint_{S+S_0} \psi\nabla\psi \, d\bar{S}$$
$$= \oint_{S\,(\text{infinity})} \psi\nabla\psi \, d\bar{S} - \oint_{S_0\,(\text{surface})} \psi\nabla\psi \, d\bar{S}$$
$$= \oint \psi\nabla\psi \, d\bar{S} - \oint_{S_0} \psi \vec{E_p} \, d\bar{S}$$
$$= \oint_S \psi\nabla\psi \, d\bar{S} - \bar{A} \oint_{S_0} \psi \, d\bar{S}$$

Potential of induced field must satisfy
$$\psi = -E_0 x \quad \text{so that} \quad \bar{E_p} = -\vec{E_0}$$
inside. Since ψ is continuous, the second integral can be written as:

$$\bar{A} \oint \psi \, d\bar{S} = -E_0 \bar{A} \oint z \, d\bar{S}$$

$$= -E_0 A V$$

The first integral goes to zero since ψ decreases with $1/r^2$. Hence

$$\int E_p^{\,2} \, dV = E_0 \cdot A \vec{V} = \frac{A^2 V}{\left(\frac{\alpha}{\varepsilon_0 V} - 1\right)}$$

velocity integral is

$$\int v^2 \, dV = \frac{u^2 V}{\left(\frac{\alpha}{\varepsilon_0 V} - 1\right)}$$

, 'Because from the boundary conditions $\bar{v}$ correlates with E_p, $\bar{u}$ correlates with $\bar{A}$'

Thus the kinetic energy of the fluid is:

$$T = \frac{1}{2} \rho \int v^2 \, dV$$

$$= \frac{1}{2} \frac{\rho V}{\frac{\alpha}{\varepsilon_0 V} - 1} u^2$$

Added mass is therefore $\boxed{m_{added} = \frac{\rho V}{\frac{\alpha}{\varepsilon_0 V} - 1}}$