Physics Cup Problem 1

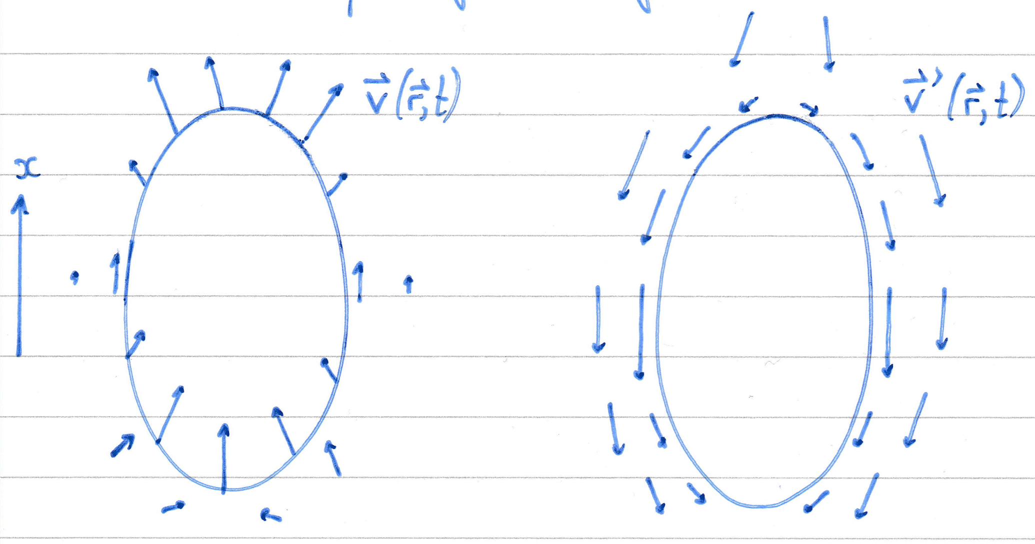
m = Mass of body
m_{eff} = Effective mass
$\Delta m = m_{eff} - m$ = Added mass

If a force of magnitude F is applied to the body, directed along the x-axis, Newton's Second Law gives:

$$F = \frac{dp_x}{dt} = m\frac{du_x}{dt} + \frac{d}{dt}\left(\rho\int_{fluid} v_x(\vec{r},t)\,dV\right) = m_{eff}\frac{du_x}{dt}$$

$$\therefore \quad \Delta m\, u_x = \rho\int_{fluid} v_x(\vec{r},t)\,dV$$ (no constant of integration as $v_x(\vec{r},t) = 0$ when $u_x = 0$)

where u_x is the speed of the body at time t.



Lab frame   Body frame

To work out the integral, we change to the frame of the body moving at speed u_x at time t. Then $\vec{v}'(\vec{r},t) = \vec{v}(\vec{r},t) - u_x\hat{x}$.

In the body frame, the fluid is still irrotational:

$$\oint \vec{v}'(\vec{r},t)\cdot d\vec{r} = \oint \vec{v}(\vec{r},t)\cdot d\vec{r} - u_x\hat{x}\cdot\oint d\vec{r} = 0.$$

2 The velocity field is determined by:

(i) Incompressible: $\oint_{\partial S} \vec{v}'(\vec{r},t) \cdot d\vec{S} = 0$ for closed surface ∂S

(ii) Irrotational: $\oint \vec{v}'(\vec{r},t) \cdot d\vec{r} = 0$ about a closed loop

(iii) Boundary at infinity: $\vec{v}'(\vec{r},t) \rightarrow -U_x \hat{x}$ as $|\vec{r}| \rightarrow \infty$

(iv) Boundary of body: $\vec{v}'(\vec{r},t) \cdot \hat{n} = 0$ at the surface of the body, where $\hat{n}$ is a unit normal.

These are exactly the defining relations for an identically shaped body with $\varepsilon = 0$ placed in a uniform electric field $\vec{E} = -E_x \hat{x}$.

The only difference is that there will be an electric field inside the body, whereas the fluid velocity is only defined outside.

In addition, we know that the body is shaped so that the field inside it is homogeneous in this situation for any dielectric constant.

The correspondence is:

$$\vec{v}'(\vec{r},t) \longleftrightarrow \vec{E}'(\vec{r},t)$$
$$U_x \longleftrightarrow E_x$$

or:

$$\vec{v}'(\vec{r},t) = c\vec{E}'(\vec{r},t)$$
$$U_x = cE_x$$

where c is a constant to ensure correct dimensions.

3 The field $\vec{E}(\vec{r},t) = \vec{E}'(\vec{r},t) + E_x\hat{x}$ is an electric field defined everywhere which tends to zero as $|\vec{r}| \to \infty$ and which is irrotational. So:

$$\int_{\text{fluid}} v_x(\vec{r},t)\, dV = \int_{\text{fluid}} v_x'(\vec{r},t) + u_x\, dV$$

$$= c\int_{\text{fluid}} E_x'(\vec{r},t) + E_x\, dV$$

$$= c\int_{\text{fluid}} E_x(\vec{r},t)\, dV$$

Homogeneous inside the body

$$= c\int_{\text{all space}} E_x(\vec{r},t)\, dV - c\int_{\text{body}} E_x(\vec{r},t)\, dV$$

$$= c\int_{z=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} \frac{\partial\varphi}{\partial x}\, dx\, dy\, dz - cE_{x\,\text{body}}V$$

$$= c\int_{z=-\infty}^{\infty}\int_{y=-\infty}^{\infty} \varphi(x\to\infty) - \varphi(x\to-\infty)\, dy\, dz - c(E_{x\,\text{body}}' + E_x)V$$

$$= -c(E_{x\,\text{body}}' + E_x)V.$$

What I'm really doing here is:

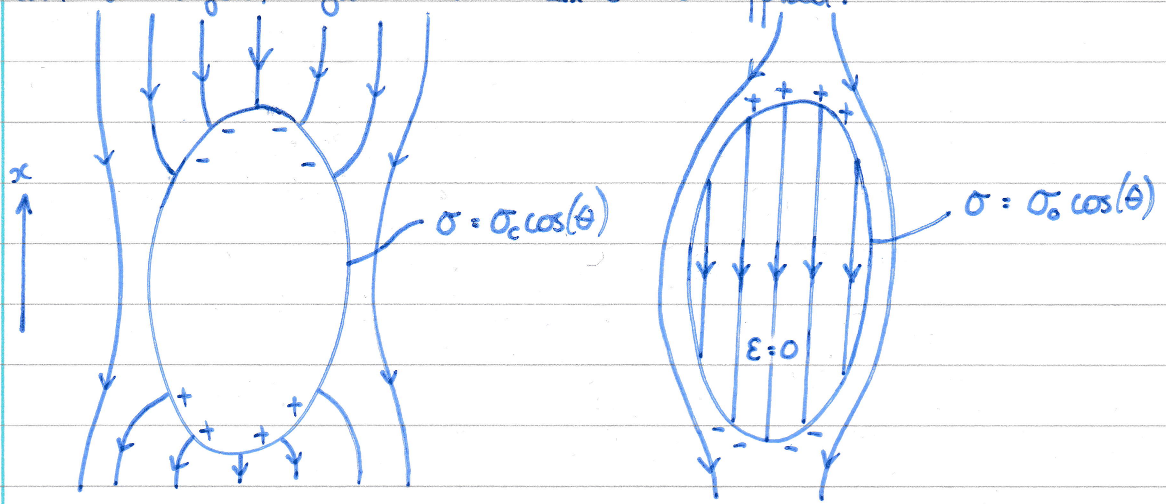
$$\int_{\text{all space}} \vec{E}\cdot\hat{x}\, dV = \int_{\text{all space}} \vec{\nabla}\varphi\cdot\hat{x}\, dV \qquad (\text{as } \vec{\nabla}\times\vec{E} = \vec{0})$$

$$= \int_{\text{all space}} \vec{\nabla}\cdot(\varphi\hat{x})\, dV$$

$$= \oint_{\text{boundary of space}} \varphi\hat{x}\cdot\vec{dS} \qquad (\text{Gauss' Theorem})$$

$= 0$ as $\varphi \to$ const. when $|\vec{r}| \to \infty$.

So, what we need is the electric field inside the body when $\varepsilon = 0$ and a uniform field $\vec{E} = -E_x \hat{x}$ is applied.



Conductor ($\varepsilon \to \infty$)

When $\varepsilon = 0$

When the body is placed in the homogeneous field the charges arrange themselves on the surface to cancel out any field inside the body (metallic $\Rightarrow$ conductor). This means they arrange themselves in a similar way to what we would get if the body was a dielectric, as this produces a homogeneous field inside.

If the body was a dielectric, the field inside would be homogeneous so the polarisation must be homogeneous. The surface charge density is:

$$\sigma = \vec{P} \cdot \hat{n} \propto \cos(\theta)$$

where θ is the angle between $\hat{n}$ and $\hat{x}$.

For the conducting case, let $\sigma = \sigma_c \cos(\theta)$.

For the $\varepsilon \to 0$ case, let $\sigma = \sigma_0 \cos(\theta)$.

5 For the conductor, the total field inside is $\vec{E}_c - E_x\hat{x} = \vec{0}$, where $\vec{E}_c$ is the field produced by the charge distribution $\sigma = \sigma_c \cos(\theta)$.

Similarly, the field inside the $\varepsilon = 0$ body is $\vec{E}_{in}' = \vec{E}_0 - E_x\hat{x}$, where $\vec{E}_0$ is the field produced by the charge distribution $\sigma = \sigma_0 \cos(\theta)$.

Since the two charge distributions are identical up to a constant:

$\frac{|\vec{E}_0|}{|\vec{E}_c|} = \frac{\sigma_0}{\sigma_c}$ (and both $\vec{E}_0$ and $\vec{E}_c$ are parallel to the x-axis)

We can work out σ_c:

$$\vec{P} = \frac{\vec{p}_{tot}}{V} = -\frac{\alpha E_x}{V}\hat{x}$$

$$\sigma = \vec{P}\cdot\hat{n} = -\frac{\alpha E_x}{V}\cos(\theta) \Longrightarrow \sigma_c = -\frac{\alpha E_x}{V}.$$

So, setting $\vec{E}_{in}' = E_{in}'\hat{x}$, $\vec{E}_0 = E_0\hat{x}$ and $\vec{E}_c = E_c\hat{x}$:

$$E_{in}' = E_0 - E_x = \frac{\sigma_0}{\sigma_c}E_c - E_x = \left(\frac{\sigma_0}{\sigma_c} - 1\right)E_x = -\left(\frac{\sigma_0 V}{\alpha E_x} + 1\right)E_x.$$

To find σ_0 we note that the normal component of the total electric field must be zero just outside the $\varepsilon = 0$ body. This is because:

$$\Delta D_\perp = 0 \Longrightarrow \varepsilon_{out}E_{\perp out} = \underset{\uparrow\,=0}{\varepsilon_{in}}E_{\perp in} = 0 \Longrightarrow E_{\perp out} = 0.$$

Now consider the highest point (most positive x) of the body on its axis of symmetry. Here, the normal must point in the x-direction by symmetry. So:

$$\Delta E_\perp = \frac{\sigma_0}{\varepsilon_0} \qquad (\text{as } \theta = 0)$$

$$\therefore \; \sigma_0 = -\varepsilon_0 E_{in}' \qquad (\text{as no } E_\perp \text{ outside}).$$

Hence:

$$E_{in}' = -\left(-\frac{\varepsilon_0 E_{in}' V}{\alpha E_x} + 1\right) E_x = \frac{\varepsilon_0 V}{\alpha} E_{in}' - E_x$$

$$\therefore \; E_{in}' = \frac{-E_x}{1 - \frac{\varepsilon_0 V}{\alpha}} .$$

So:

$$\int_{\text{fluid}} v_x(\vec{r}, t)\, dV = -c\left(\frac{-E_x}{1-\frac{\varepsilon_0 V}{\alpha}} + E_x\right)V = cE_x\left(\frac{\frac{\varepsilon_0 V}{\alpha}}{1-\frac{\varepsilon_0 V}{\alpha}}\right)V = \frac{u_x V}{\frac{\alpha}{\varepsilon_0 V} - 1}$$

$$\therefore \; \boxed{\Delta m = \frac{\rho V}{\frac{\alpha}{\varepsilon_0 V} - 1}}$$