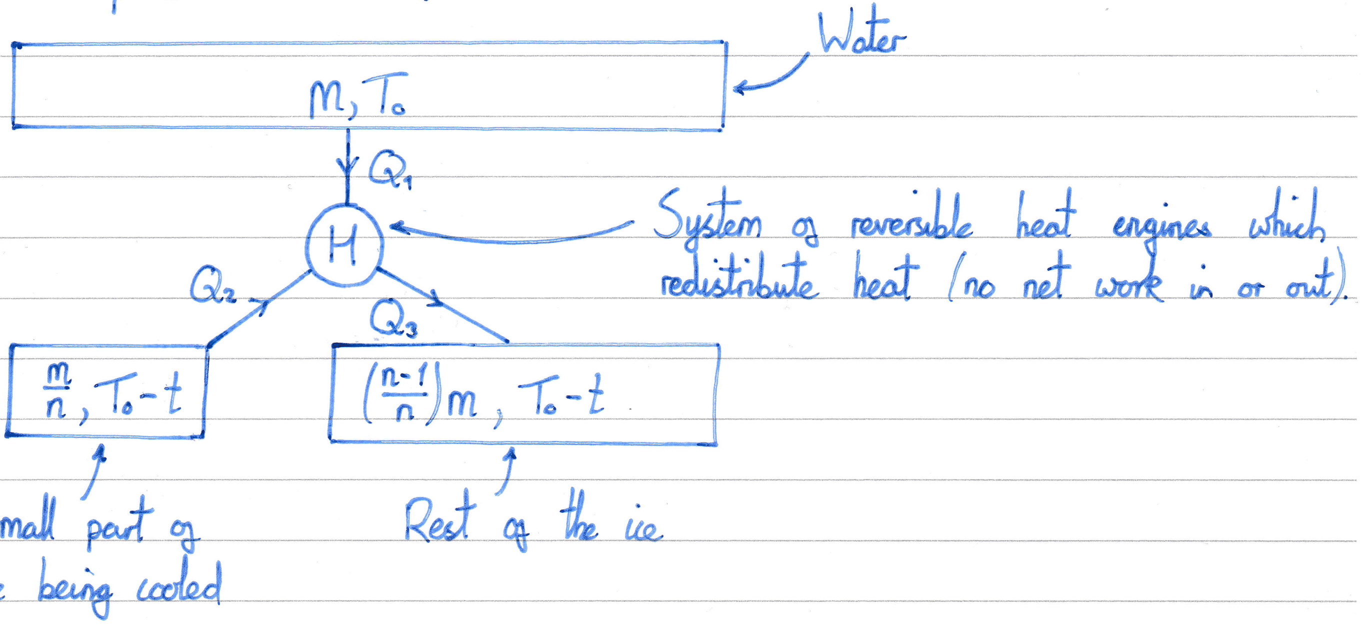
Physics Cup Problem 2

We will first derive a formula for the lowest attainable temperature given any system of heat engines (all reversible and with negligible heat capacity) connecting parts of the ice/water system in an arbitrary way. Then we will show that this lower bound can be reached using the two heat engines operating in the directions given in the problem statement.



In the final state, the whole system except the part being cooled must have the same temperature because any temperature difference could be used to power a heat engine and achieve further cooling.

Assume that the part of the ice being cooled reaches its lowest temperature before the water freezes - this assumption will be justified a posteriori.

For small block:

$$dQ = \frac{mc_v}{n}\,dT = \frac{m\alpha T}{n}\,dT \qquad (dQ \text{ is heat added})$$

$$\therefore \quad Q_2 = -\frac{m\alpha}{n}\int_{T_0-t}^{T} T'\,dT' = \frac{m\alpha}{2n}\left(T_0^2 - 2T_0 t + t^2 - T^2\right) \qquad (1)$$

For the rest of the ice:

$$dQ = \frac{m(n-1)}{n} C_v \, dT = \frac{m\alpha(n-1)}{n} dT$$

$$\therefore \quad Q_3 = \frac{m\alpha(n-1)}{n} \int_{T_0-t}^{T_0} T' \, dT' = \frac{m\alpha(n-1)}{2n} \left(2T_0 t - t^2\right) \qquad ②$$

Conservation of energy: $Q_1 + Q_2 = Q_3$ ③

Second Law of Thermodynamics:

$$\Delta S = -\frac{Q_1}{T_0} + \frac{m\alpha}{n} \int_{T_0-t}^{T} dT' + \frac{m\alpha(n-1)}{n} \int_{T_0-t}^{T_0} dT' \geq 0$$

$$\therefore \quad Q_1 \leq \frac{m\alpha}{n}\left(T_0 T - T_0^2 + T_0 t\right) + \frac{m\alpha(n-1)}{n}\left(T_0 t\right) \qquad ④$$

Combining ①, ②, ③ and ④:

$$\frac{m\alpha(n-1)}{2n}(2T_0 t - t^2) - \frac{m\alpha}{2n}\left(T_0^2 - 2T_0 t + t^2 - T^2\right) \leq \frac{m\alpha}{n}\left(T_0 T - T_0^2 + T_0 t\right) + \frac{m\alpha(n-1)}{n} T_0 t$$

$$\therefore \quad \cancel{2nT_0 t} - nt^2 - \cancel{2T_0 t} + \cancel{t^2} - T_0^2 + \cancel{2T_0 t} - \cancel{t^2} + T^2 \leq 2T_0 T - 2T_0^2 + \cancel{2T_0 t} + \cancel{2nT_0 t} - \cancel{2T_0 t}$$

$$\therefore \quad T_0^2 - 2T_0 T + T^2 \leq nt^2$$

$$\therefore \quad (T_0 - T)^2 \leq nt^2$$

$$\therefore \quad T \geq T_0 - t\sqrt{n} \qquad (\text{as } T < T_0).$$

The minimum possible temperature is $T = T_0 - t\sqrt{n}$. This is achieved when all the heat engines are reversible and there is no thermal contact between parts of the system at different temperatures except through the heat engines.

We still need to show the water doesn't freeze:

$$Q_1 = Q_3 - Q_2$$

$$= \frac{m\alpha(n-1)}{2n}\left(2T_0t - t^2\right) - \frac{m\alpha}{2n}\left(T_0^2 - 2T_0t + t^2 - T_0^2 + 2\sqrt{n}\,T_0t - nt^2\right)$$

$$= \frac{m\alpha}{2n}\left(2nT_0t - \cancel{nt^2} - \cancel{2T_0t} + \cancel{t^2} - \cancel{T_0^2} + \cancel{2T_0t} - \cancel{t^2} + \cancel{T_0^2} - 2\sqrt{n}\,T_0t + \cancel{nt^2}\right)$$

$$= m\alpha T_0 t\left(1 - \frac{1}{\sqrt{n}}\right).$$

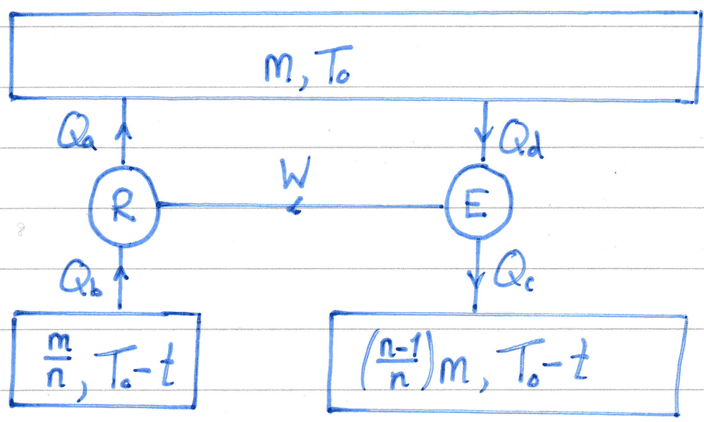
The water freezes if $Q_1 > m\lambda$ or:

$$\frac{\lambda}{\alpha T_0 t} < \left(1 - \frac{1}{\sqrt{n}}\right)$$

This is not the case because if $t \approx 1K$ then:

$$\frac{\lambda}{\alpha T_0 t} \approx 162.8$$

The following setup can achieve the minimum temperature using a reversible engine and refrigerator:



$$Q_b = Q_3$$
$$Q_c = Q_2$$
$$Q_d - Q_a = Q_1$$

So, the lower bound is attainable:

$$T = T_0 - t\sqrt{n}.$$