Physics Cup 2019, Problem 1

In order to solve this problem, we will first solve an analog electromagnetic problem, and then, apply the necessary changes in order to obtain the result for the hydrodynamics problem.

We start our analogy by realizing that in the hydrodynamics problem, we wish to solve the following equations for the velocity field of the liquid **outside the contour of the object:**

$$\nabla \cdot \vec{v} = 0$$

$$\nabla \times \vec{v} = 0$$

We notice that these equations are very similar to two of the Maxwell equations in a space with charge density 0:

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \times \vec{E} = 0$$

We can, therefore, think of the object immerse in a liquid the same way as we think of a dielectric corpse immerse on an electric field.

We will work on the object's frame, and, therefore, will have to apply some corrections further down the resolution.

As we have a physical boundary that prevents the water from entering the object, we can estipulate that our analog electromagnetic problem shall have a boundary condition for the electric field perpendicular to the surface of the object:

$$E_p = 0$$

Where E_p denotes the electric field perpendicular to the surface at the boundary of the object.

By using this boundary condition, we can arrive at the conclusion that the dielectrics constant of our analog dielectric object is $\epsilon = 0$, since the perpendicular displacement vector $\vec{D} = \epsilon\epsilon_0\vec{E}$ must be continuous on the boundary of the object, a dieletric constant equal to 0 ensures that the perpendicular $\vec{D}$ is 0 inside the object, and thus, ensures that the perpendicular $\vec{E}$ is 0 outside the object.

$$\epsilon = 0$$

To calculate the force acting on the object, we will integrate the force due to the different pressure of the liquid at different points. Using the unsteady Bernoulli equation (derivation in appendix 1), we are concerned only with the pressure difference relative to the unsteady character of the flow, (since neither of the other two terms involve the acceleration of the body and added mass is a phenomenon related to the acceleration):

$$\Delta P = \rho \frac{\partial \varphi}{\partial t}$$

Where φ denotes the velocity potential, which is analog to the electric potential.

We must integrate this pressure difference over the surface of our object that is perpendicular to $\overrightarrow{E_0}$ to obtain the force acting on the x direction.

As the shape of the field lines (electric field lines which are analog to velocity field lines) does not change in time, we can think of our electric potential as a spatial function that does not vary in time, times $\overrightarrow{E_0}$:

$$\varphi(x, y, z, t) = E_0(t) * \delta(x, y, z)$$

The rate of change of the function φ with respect to time is:

$$\frac{\partial \varphi}{\partial t} = \frac{dE_0}{dt} * \ \delta(x, y, z)$$

Hence, we can write:

$$\frac{\partial \varphi}{\partial t} = \frac{\frac{dE_0}{dt} * \varphi}{E_0}$$

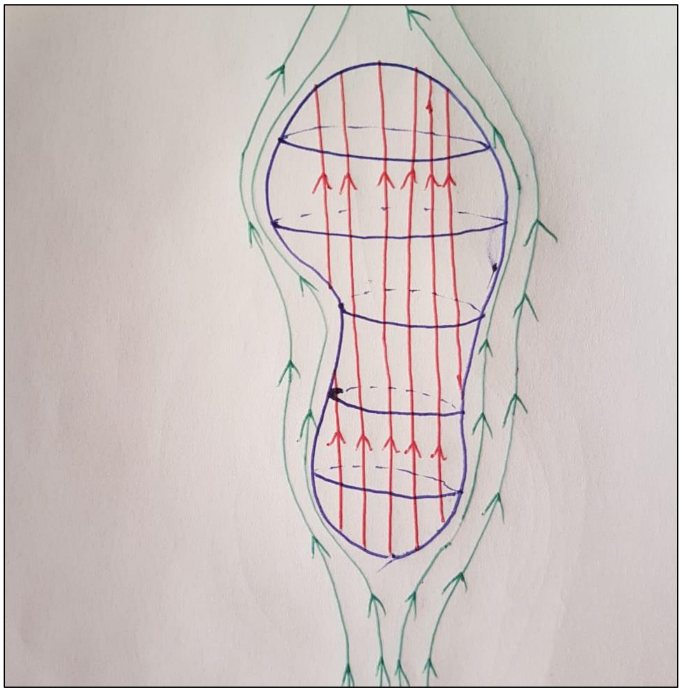
The force is obtained by integrating ΔP across the surface perpendicular to $\overrightarrow{E_0}$:

$$F = \rho \frac{\frac{dE_0}{dt}}{E_0} \iint_{S_p} \varphi * dS_p$$

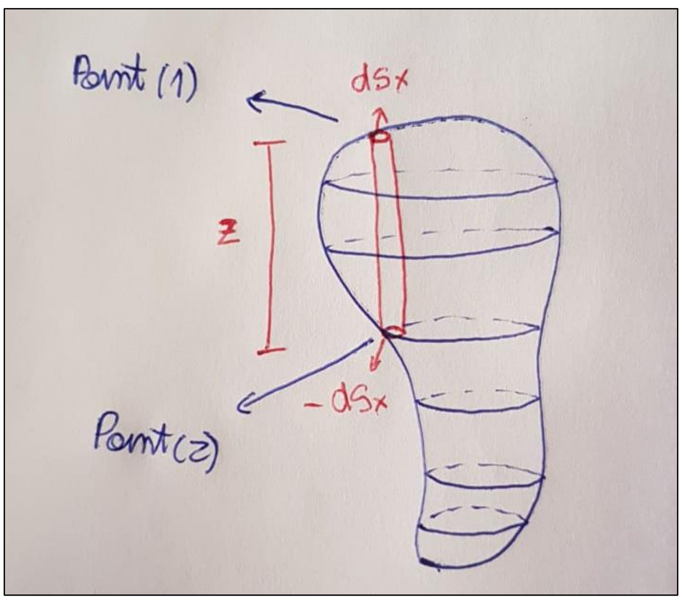
Here, S_p denotes the surface of the object that is perpendicular to $\overrightarrow{E_0}$.

As we don't know neither the shape of our object nor the potential function, we will have to use an unusual procedure in order to calculate the integral above. We will take advantage of the fact that the electric field inside the dielectric is uniform.

Let our object be of a generic shape. We can draw the field lines on the interior and exterior as follows:



We can divide our object in small columns as shown:



We can then write:

$$dF = \rho \frac{\frac{dE_0}{dt}}{E_0} * \left(\varphi(point\ 1) - \varphi(point\ 2)\right) dS_x$$

Hence:

$$dF = \rho \frac{\frac{dE_0}{dt}}{E_0} \Delta\varphi_{12} dS_x$$

As the electric field inside the object is uniform, we can write the potential difference between points 1 and 2 as a function of the electric field inside the dielectric, however, we notice that we have to apply a correction due to the fact that we are not in the reference frame in which the water is not moving far from the object:

$$\Delta\varphi_{12} = z(E_{in} - E_0)$$

Where E_{in} denotes the electric field inside the object, and the term E_0 represents the correction that was mentioned, which is "simulated" by adding an electric field E_0 at all points, "emulating" the effect of the change of reference frame.

Substituting this value of $\Delta\varphi$ into the equation for dF, we obtain:

$$dF = \rho \frac{\frac{dE_0}{dt}}{E_0} (E_{in} - E_0) z dS_x$$

We notice that zdS_x denotes an infinitesimal element of volume, and, therefore:

$$dF = \rho \frac{\frac{dE_0}{dt}}{E_0} (E_{in} - E_0) dV$$

$$F = \int \rho \frac{\frac{dE_0}{dt}}{E_0} (E_{in} - E_0) dV = \rho \frac{\frac{dE_0}{dt}}{E_0} (E_{in} - E_0) V$$

Now, we have to determine E_{in} in order to obtain the force.

First, we can write the following equation relating the polarization, volume and polarizability of our object:

$$\vec{P}V = \alpha \overrightarrow{E_0}$$

This relation arises from the fact that we can think of the polarized object as a similar object made of positive charges slightly dislocated from another object made of negative charges. This charge distribution is seen a simple electric dipole, when observed from a far enough point.

Writing the polarization in terms of the internal electric field:

$$E_{in}(\epsilon - 1)\epsilon_0 V = \alpha_\epsilon E_0$$

$$E_{in} = \frac{\alpha_\epsilon E_0}{(\epsilon - 1)\epsilon_0 V}$$

Here, α_ϵ denotes the polarization as a function of the dielectric constant ϵ.

As was already explained, we in our analog problem, we take the dielectric constant to be 0, therefore, the internal field is:

$$E_{in} = \frac{\alpha_0 E_0}{(\epsilon - 1)\epsilon_0 V}$$

Substituting the expression for E_{in} into the force equation yields:

$$F = \rho \frac{\frac{dE_0}{dt}}{E_0} \left(\frac{\alpha_0 E_0}{(\epsilon - 1)\epsilon_0 V} - E_0 \right) V$$

$$F = \rho \frac{dE_0}{dt} \left(\frac{\alpha_0}{(\epsilon - 1)\epsilon_0 V} - 1 \right) V$$

However, we have a problem because the problem statement only gives us the polarizability of a metal object (we assume the dielectric constant of the metal to be $\epsilon = \infty$), whereas we need the polarizability of an object of the same shape, but with $\epsilon = 0$. We have to find a way to obtain the polarizability of an object of the same shape as the metal one, as a function of the polarizability of the metal object.

To do this, we can start by start by writing down a relationship between the internal electric field caused by the polarized object, and the polarization of the object as:

$$E_{in}' = -\frac{P}{C\epsilon_0}$$

Here, E_{in}' denotes the internal electric field generated by the polarized object and C is a constant related to the shape of the object.

We can write this because, as the total electric field inside the object is uniform, we can conclude that the internal electrical field caused by the polarized object is also uniform, since the total internal electric field is obtained as the superposition of E_0 and E_{in}'.

Using $P = (\epsilon - 1)\epsilon_0 E_{in}$, we can write:

$$E_{in} = E_0 + E_{in}' = E_0 - \frac{(\epsilon - 1)E_{in}}{C}$$

Hence:

$$E_{in} = \frac{CE_0}{(C - 1) + \epsilon}$$

The polarization is:

$$P = \frac{CE_0(\epsilon - 1)\epsilon_0}{(C - 1) + \epsilon}$$

Using $\vec{P}V = \alpha \overrightarrow{E_0}$ once more, we obtain the polarizability as:

$$\alpha_\epsilon = C\epsilon_0 V \left(\frac{\epsilon - 1}{(C - 1) + \epsilon} \right)$$

Now, we notice that as the value of ϵ approaches infinity, the expression for α_ϵ reduces to:

$$\alpha_\infty \approx C\epsilon_0 V$$

Hence, the C constant is given by:

$$C = \frac{\alpha_\infty}{\epsilon_0 V}$$

Substituting this value of C into the expression for α_ϵ:

$$\alpha_\epsilon = \frac{\alpha_\infty}{\epsilon_0 V} \epsilon_0 V \left(\frac{\epsilon - 1}{\left(\frac{\alpha_\infty}{\epsilon_0 V} - 1 \right) + \epsilon} \right)$$

Hence, α_0 is:

$$\alpha_0 = \alpha_\infty \left(\frac{-1}{\left(\frac{\alpha_\infty}{\epsilon_0 V} - 1 \right)} \right)$$

Finally, inserting this value of α_0 into the expression for F and setting $\epsilon = 0$, we obtain:

$$F = \rho \frac{dE_0}{dt} \left(\frac{\alpha_\infty \left(\frac{-1}{\left(\frac{\alpha_\infty}{\epsilon_0 V} - 1 \right)} \right) \alpha_0}{(-1)\epsilon_0 V} - 1 \right) V$$

$$F = \rho \frac{dE_0}{dt} \left(\frac{\alpha_\infty}{\alpha_\infty - \epsilon_0 V} - 1 \right) V$$

Changing $\frac{dE_0}{dt} \longrightarrow \frac{dv}{dt}$ (after completing the analog problem, we have to substitute back v instead of E), we obtain the added mass as:

$$m_+ = \frac{F}{\frac{dv}{dt}} = \rho \left(\frac{\alpha_\infty}{\alpha_\infty - \epsilon_0 V} - 1 \right) V$$

Appendix 1: derivation of unsteady Bernoulli equation

Refference:
http://web.mit.edu/2.016/www/handouts/Unsteady_Bernoulli%27s_Derivation_050921.pdf

We start with:

$$\vec{F} = m\vec{a}$$

Hence:

$$\rho \vec{a} = \rho \frac{D\vec{v}}{Dt}$$

We can write $\frac{D\vec{v}}{Dt}$ as:

$$\frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + \frac{\partial\vec{v}}{\partial x} * \frac{\partial x}{\partial t} + \frac{\partial\vec{v}}{\partial y} * \frac{\partial y}{\partial t} + \frac{\partial\vec{v}}{\partial z} * \frac{\partial z}{\partial t}$$

$$\frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + \left(\left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\right)$$

As the flow is considered irrotational, we can define a velocity potential such that $\nabla\varphi = \vec{v}$.

Writing $\frac{D\vec{v}}{Dt}$ in terms of the velocity potential and equating it with the terms relative to the forces that act on the system, we are left with:

$$\rho\left(\frac{\partial\nabla\varphi}{\partial t} + \nabla\left(\frac{\nabla\varphi \cdot \nabla\varphi}{2}\right)\right) = -\nabla p - \rho g$$

Hence:

$$\rho\left(\frac{\partial\varphi}{\partial t} + \left(\frac{\nabla\varphi \cdot \nabla\varphi}{2}\right)\right) + p + \rho g z = C$$